Non-markovian global persistence in phase-ordering kinetics

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Contents :

I. Ageing phenomena in simple magnets

consider a simple magnet (ferromagnet, i.e. Ising model, non-conserved dynamics)

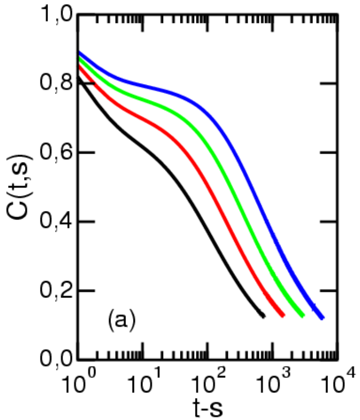
1. prepare system initially at high temperature $T \gg T_c > 0$
2. quench to temperature $T < T_c \rightarrow$ phase-ordering kinetics
   (or $T = T_c \rightarrow$ nonequilibrium critical dynamics)
   $\rightarrow$ **non-equilibrium state**
3. fix T and observe dynamics

formation of ordered domains, of linear size $L = L(t) \sim t^{1/\mathbf{z}}$
dynamical exponent z

Criteria for **physical ageing** :

1. slow (i.e. non-exponential dynamics)
2. breaking of time-translation-invariance
3. dynamical scaling

Example for ageing : $3D$ Glauber-Ising model, $T < T_c$



slow dynamics (non-exponential)

dynamical scaling

no time-translation invariance

$C(t,s)$: autocorrelation function, quenched to $T < T_c$

scaling regime : $t, s \gg \tau_{\rm micro}$ and $t - s \gg \tau_{\rm micro}$

Scaling behaviour & exponents

single relevant time-dependent length scale $L(t) \sim t^{1/z}$

Bray 94, Janssen et al. 92, Cugliandolo & Kurchan 90s, Godrèche & Luck 00, ...

$\phi(t, \mathbf{r})$ – space-time-dependent order-parameter (magnetisation)

$$\text{correlator } C(t,s;\mathbf{r}) := \langle \phi(t,\mathbf{r})\phi(s,\mathbf{0})\rangle = s^{-b} f_C(t/s, |\mathbf{r}|^z/(t-s))$$

$$\text{response } R(t,s;\mathbf{r}) := \left.\frac{\delta\langle\phi(t,\mathbf{r})\rangle}{\delta h(s,\mathbf{0})}\right|_{h=0} = s^{-1-a} f_R(t/s, |\mathbf{r}|^z/(t-s))$$

No fluctuation-dissipation theorem : $\boxed{R(t,s;\mathbf{r}) \neq T\partial C(t,s;\mathbf{r})/\partial s}$

values of exponents : equilibrium correlator $\to$ classes **S** and **L**

$$C_{\rm eq}(\mathbf{r}) \sim \left\{ \begin{array}{l} \exp(-|\mathbf{r}|/\xi) \\ |\mathbf{r}|^{-(d-2+\eta)} \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} \text{class } \mathbf{S} \\ \text{class } \mathbf{L} \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} a = 1/z \\ a = (d-2+\eta)/z \end{array} \right.$$

if $T < T_c$: $z = 2$ and $b = 0$
for $y \to \infty$: $f_{C,R}(y) \sim y^{-\lambda_{C,R}/z}$,

if $T = T_c$: $z = z_c$ and $b = a$
$\lambda_{C,R}$ independent exponents

II. Persistence probability

consider a different kind of observable : BRAY, DERRIDA, GODRÈCHE, ... 94

probability that magnetisation has *not* changed sign up to time t ?

here : **global order-parameter** in a volume Ω

$$\widehat{\phi}_{\mathbf{0}}(t) := \frac{1}{|\Omega|^{1/2}} \int_{\Omega} \mathrm{d}\mathbf{r}\, \phi(t,\mathbf{r})$$

study the **global persistence probability** $P_{\rm g}(t)$
if natural dynamical scaling, expect for $t \to \infty$

$$P_{\rm g}(t) \sim t^{-\theta_g} \quad , \quad \boxed{\theta_g = \textbf{global persistence exponent}}$$

MAJUMDAR, BRAY, CORNELL, SIRE 96

may also consider dependence of block size
(**block persistence**) CUEILLE & SIRE 96/97

III. Relationship with Markov processes

aim : derive a scaling relation for the global persistence exponent θ_g, for $T \leq T_c$, for **Markov processes**

argument proceeds in **5** steps MAJUMDAR ET AL. 96, CUEILLE & SIRE 96/97

1. :

after quench, domains correlated up to linear size $L(t) \ll |\Omega|^{1/d}$, system consists of (almost) uncorrelated domains, linear size $L(t)$.
$\Longrightarrow$ $\widehat{\phi}_{\mathbf{0}}(t)$ is sum over uncorrelated random variables, with **finite** moments for each finite time t

$$(i) \ \ \langle \widehat{\phi}_{\mathbf{0}}(t) \rangle = 0 \ \ , \ \ (ii) \ \ \langle \widehat{\phi}_{\mathbf{0}}^2(t) \rangle \sim L(t)^{d-bz}$$

apply **central limit theorem** $\Longrightarrow$ $\boxed{\widehat{\phi}_{\mathbf{0}}(t) \text{ gaussian process}}$ $\forall t < \infty$

$\underline{\textbf{2.}}$: scaling analysis for $|\Omega| \to \infty$ and $t_1 > t_2$

$$
\begin{aligned}
\langle \widehat{\phi}_{\mathbf{0}}(t_1)\widehat{\phi}_{\mathbf{0}}(t_2)\rangle &= \lim_{\mathbf{k}\to\mathbf{0}} \langle \widehat{\phi}_{\mathbf{k}}(t_1)\widehat{\phi}_{-\mathbf{k}}(t_2)\rangle \\
&= \lim_{\mathbf{k}\to\mathbf{0}} \frac{1}{|\Omega|}\int_{\Omega^2} \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2\; e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)}\; \langle \phi(t_1,\mathbf{r}_1)\phi(t_2,\mathbf{r}_2)\rangle \\
&= \lim_{\mathbf{k}\to\mathbf{0}} \frac{1}{|\Omega|}\int_{\Omega^2} \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2\; e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}\; t_2^{-b} f_C\left(\frac{t_1}{t_2}, \frac{\mathbf{r}_1-\mathbf{r}_2}{L(t_1-t_2)}\right) \\
&= \lim_{\mathbf{k}\to\mathbf{0}} \int_{\Omega} \mathrm{d}\mathbf{r}\; e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\; t_2^{-b} f_C\left(\frac{t_1}{t_2}, \frac{\mathbf{r}}{L(t_1-t_2)}\right) \\
&= t_2^{(d-bz)/z}\left(\frac{t_1}{t_2}-1\right)^{d/z} \int_{\Omega} \mathrm{d}\mathbf{r}\; f_C\left(\frac{t_1}{t_2}, \mathbf{r}\right) \\
&= t_2^{(d-bz)/z}\hat{f}\left(\frac{t_1}{t_2}\right) \quad ; \quad \text{where } \hat{f}(y) \stackrel{y\to\infty}{\sim} y^{(d-\lambda_C)/z}.
\end{aligned}
$$

3. : define the **normalised autocorrelator**

$$N(t_1, t_2) := \frac{\langle \widehat{\phi}_{\mathbf{0}}(t_1)\widehat{\phi}_{\mathbf{0}}(t_2)\rangle}{\sqrt{\langle \widehat{\phi}_{\mathbf{0}}^2(t_1)\rangle\langle \widehat{\phi}_{\mathbf{0}}^2(t_2)\rangle}} = \hat{f}_N\left(\frac{t_1}{t_2}\right)$$

$\Longrightarrow$ asymptotics for $y \to \infty$: $\hat{f}_N(y) \sim y^{(d-2\lambda_C+bz)/(2z)}$
New time variable $\mathbf{T} = \ln \mathbf{t}$, find

$$N(t_1, t_2) = \bar{N}(T_1, T_2) = n(T_1 - T_2)$$

$\Longrightarrow$ **the gaussian process describing $\widehat{\phi}_{\mathbf{0}}(T)$ is stationary.**

Asymptotics for $T \to \infty$:

$$\boxed{n(T) \sim e^{-\mu T} \ \ , \ \ \mu = (2\lambda_C - d - bz)/(2z)}$$

4. : **Lemma 1 :** (Doob 1942) *A gaussian, stationary stochastic process $X(t)$ with $\langle X(t)\rangle = 0$ is markovian, if and only if the autocorrelator has exactly an exponential form*

$$\langle X(t_1)X(t_2)\rangle = X_0\, e^{-\mu|t_1-t_2|}.$$

where μ is a constant and X_0 a normalisation.
Conclusion : if $\widehat{\phi}_{\mathbf{0}}(T)$ is markovian, one must have exactly $n(T) = e^{-\mu T}$, with $\mu = (2\lambda_C - d - bz)/z$.
5. : **Lemma 2 :** (Slepian 1962) *Consider a gaussian and stationary stochastic process with an autocorrelator $\langle X(T)X(0)\rangle = e^{-\mu|T|}$. Then the global persistence probability for $X(t)$ is given by*

$$P_{\rm g}(T) = \frac{2}{\pi}\arcsin\left(e^{-\mu T}\right).$$

Conclusion : for $T\to\infty$, and $t_2 \to 1$, find $P_{\rm g}(T) \sim e^{-\mu T} \sim t_1^{-\mu}$

Scaling relation for θ_g

long times : $P_{\rm g}(t) \sim t^{-\theta_g}$, with

$$\theta_g = \mu = \left(2\lambda_C - d - bz\right)/\left(2z\right)$$

Provided $\widehat{\phi}_{\mathbf{0}}(t)$ is a Markov process, have scaling relations :
a) **non-equilibrium critical dynamics** $T = T_c$, $b = (d-2+\eta)/z$

$$\theta_g z = \lambda_C - d + 1 - \frac{1}{2}\eta$$

b) **phase-ordering kinetics :** $\mathbf{T} < \mathbf{T_c}$, $z=2$, $b=0$

$$\theta_g z = \lambda_C - \frac{d}{2} \geq 0$$

Use these scaling relations to **test the Markov property** !

Test of the markovian relation $\theta_g z = \lambda_C - d + 1 - \eta/2$, $\boxed{T = T_c}$

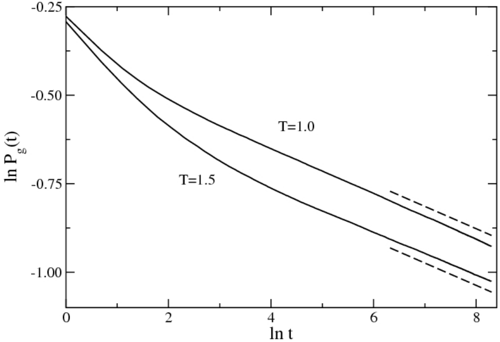
$\Longrightarrow$ generic **non-markovian** dynamics of global magnetisation

RESULTS FROM MANY DIFFERENT GROUPS 96-09

model	d	z	λ_C	η	θ_g Markov	θ_g numeric
Ising	1	2	1	1	1/4	1/4
	2	2.1667	1.588	1/4	0.214(1)	0.237(3)
					0.214(1)	0.235(5)
	3	2.043	2.78	0.0364	0.374(1)	0.41(2)
Potts-3	2	2.197	1.836	4/15	0.321(2)	0.350(2)
Potts-4	2	2.293	2.15	1/4	0.43(1)	0.474(7)
Blume-Capel	2	2.215	3.17	3/80	0.97(2)	1.080(4)
diluted Ising	3	2.62	2.75	0.037	0.28(2)	0.35(1)
double exchange	3	1.975	2.05	0.0375	0.017	0.335(9)
spherical	< 4	2	$\frac{3}{2}d - 2$	0	$(d-2)/4$	$(d-2)/4$
mean-field	> 4	2	d	0	1/2	1/2
NEKIM	1	1.75	1.51	1	0.58(1)	0.67(1)

IV. Numerical results – phase-ordering

consider $2D$ **Glauber-Ising model**, $T_c \simeq 2.27$, quench to $\mathbf{T < T_c}$.
Lattice 400×400, average over $8 \cdot 10^4$ initial configurations/noise



find two regimes of power-law decay :

1. for large times, $\theta_g = 0.063(2)$ – averaged over all values of T
2. for short times, effectively critical as long as $L(t) \ll \xi_{\rm therm}$
   estimates $\theta_g(1.8) \approx 0.18$, $\theta_g(2.0) \approx 0.20$; $\theta_g(T_c) = 0.236(3)$

Test of the markovian relation $\theta_g z = \lambda_C - d/2$, **$T < T_c$**

$\Longrightarrow$ generic **non-markovian** dynamics of global magnetisation

model	d		λ_C	θ_g Markov	θ_g numeric	
Ising	1	$T=0$	1	$1/4$	$1/4$	M96
Ising	2	$T=0$	1.24(2)	0.12(1)	$\simeq 0.09$	CS97
		$T=1.0$	1.24(2)	0.12(1)	0.062(2)	
		$T=1.5$	1.24(2)	0.12(1)	0.065(2)	
TDGL	2	$T=0$	1.24	0.12	$\simeq 0.06$	CS97
spherical	>2	$T<T_c$	$d/2$	0	0	
spherical, long-range	$>\sigma$	$T<T_c$	$d/2$	0	0	

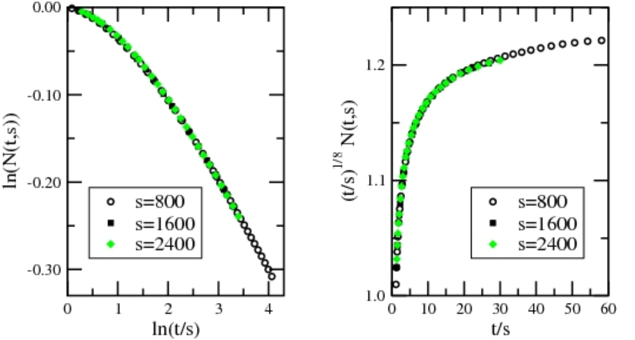
θ_g temperature-independent $\Rightarrow$ confirms that $T < T_c$ irrelevant

observation : $\left\{ \begin{array}{lll} \theta_g \geq \theta_g^{\rm mark} & ; & \text{if } T = T_c \\ \theta_g \leq \theta_g^{\rm mark} & ; & \text{if } 0 < T < T_c \end{array} \right\}$ why ?

M96 = Majumdar et al. 96 ; CS97 = Cueille & Sire 97

Form of normalised global autocorrelator $N(t,s)$

$2D$ Glauber-Ising model, quenched to $\mathbf{T} = \mathbf{1.5} < \mathbf{T_c}$



* find dynamical scaling

* observe **very long transient** towards expected asymptotics
(effective exponent $\approx$ **0.115**, expected 0.125)

* $(t/s)^{1/8} N(t,s)$ is **not** a constant

$\Longrightarrow$ **incompatible with Doob's lemma for a Markov process**

V. Conclusions

1. study long-time behaviour of *global* persistence $P_{\mathrm{g}}(t) \sim t^{-\theta_g}$
2. if Markov process for global order-parameter, then
   - $\theta_g z = \lambda_C - d + 1 - \eta/2$ at criticality $\underline{T = T_c}$
   - $\theta_g z = \lambda_C - d/2$ at low temperatures $\underline{T < T_c}$
3. satisfied in certain solvable models
   ($1D$ Glauber-Ising, spherical,...)
4. in general broken $\Longrightarrow$ **non-markovian dynamics**
   for *global* order-parameter, independently of value of z

Some open questions :

- find exactly solvable non-markovian, microscopically local, dynamics
- renormalised eqs. of motion non-local in time *and* space
  $\Longrightarrow$ what about dynamical symmetries ?

Vol. 2 – co-author M. **Pleimling** – will treat ageing phenomena in simple magnets and LSI (hopefully finished still in 2009)

MECO conference in Pont-à-Mousson (Lorraine)

The next **MECO conference** will be held in the historical abbey of the Prémontrés in Pont-à-Mousson, about in the middle between Nancy & Metz, Lorraine (France).

Dates : monday the 15$^{\text{th}}$ of march 2010 (arrival)
to friday the 19$^{\text{th}}$ of march 2010 (departure).
Web site : `http ://www.ijl.nancy-universite.fr/meco35`

Inscriptions are already open !

The organising comittee
(GPS - DPMM-IJL, CNRS – **Nancy** Université – UPVM).