Extended Scaling in High Dimensions

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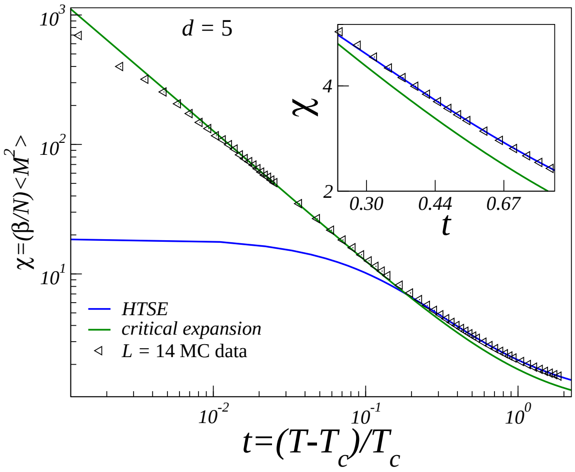
Outline

1. What is Extended Scaling ?
2. The Ising model above $d_c = 4$
3. The results

Generalities



Generalities

- Series expansion :
  Corrective terms to the leading critical behaviour :
  $$\chi \sim t^{-\gamma} \left(1 + a_1 t^{\theta} + a_2 t^{2\theta} + \cdots + b_1 t + b_2 t^2 \dots\right)$$
  $d = 5$ and $d = 6$: Guttmann (*J. Phys. A* 1981)
- High Temperature Series expansion (HTSE)
  15^{th} order for the Ising model in function of d :
  Gofman *et al* (*J. Stat. Phys.* 1993)

Extended Scaling

First step :

- Other scaling variable than $t = (T - T_c)/T_c$:

$$\tau = \frac{T - T_c}{T}$$

- Both variables vanish at the critical point :

$$\begin{aligned} \lim_{T \to T_c} t &= 0 \\ \lim_{T \to T_c} \tau &= 0, \end{aligned}$$

- but they have different high-temperature limit :

$$\begin{aligned} \lim_{T \to \infty} t &= \infty \\ \lim_{T \to \infty} \tau &= 1 \end{aligned}$$

Extended Scaling

Second step :

- High-Temp. limit of $\mathcal{O}$:

$$\lim_{T\to\infty} \mathcal{O}(T) \sim \beta^{\psi}.$$

- $\mathcal{O}$ has a temperature-dependant amplitude :

$$\mathcal{O} \sim \beta^{\psi}\tau^{-\rho}$$

- Critical behaviour of $\mathcal{O}(\tau)$ satisfies both limits when $\tau \to 0$ and $\tau \to 1$.

Extended Scaling

Extended scaling for the Ising model :

- I. Campbell, K. Hukushima and H. Takayama (*PRL* 2006 ; *PRB* 2007) :
  2d and 3d Ising model, 3d bimodal spin-glass, 3d XY and Heisenberg model.
- I. Campbell and P. Butera (*PRB* 2008) :
  Mean-Field Ising model ($d = \infty$), 2d Ising model again.
- H. Katzgraber, I. Campbell and A. Hartmann (*ArXiv* 2008) :
  1d Ising model, 2d fully-frustrated Ising model

The ϕ^4 model

- Same universality class as the Ising model.

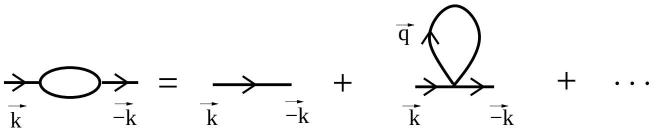
$$H_{\phi^4}[\phi] = \int d^d\vec{r} \left(\frac{1}{2} \left(\vec{\nabla}\phi(\vec{r}) \right)^2 + \frac{m^2}{2}\phi^2(\vec{r}) + \frac{g}{4!}\phi^4(\vec{r}) \right)$$

- For $d > 4$, $g \to 0$ but perturbation induced by the ϕ^4 term.

$$\exp\left(-H_{\phi^4}[\phi]\right) \approx \left(1 - \frac{g}{4!} \int d^d\vec{r}\phi^4(\vec{r}) + \dots \right) \exp\left(-H_G[\phi]\right)$$

where $H_G[\phi]$ is a gaussian hamiltonian.

The ϕ^4 model



$$\begin{aligned}
\tilde{C}(\vec{k}) &\approx \tilde{C}_G(\vec{k}) - \frac{g}{2}\tilde{C}_G(\vec{k}) \underbrace{\int \frac{d^d\vec{q}}{(2\pi)^d}\frac{1}{q^2+m^2}}_{\sim m^{d-2}} \tilde{C}_G(\vec{k}) + \dots \\
&\approx \tilde{C}_G(\vec{k})\left(1 - \frac{g}{2}m^{d-2}\tilde{C}_G(\vec{k}) + \dots\right)
\end{aligned}$$

The ϕ^4 model

$$\chi/\beta \sim \tilde{C}(\vec{0}) \approx \chi_G/\beta \left(a_0 + a_1 m^{d-4} + \dots\right)$$

where $\chi_G/\beta \sim \tilde{C}_G(\vec{0}) \sim m^{-2}$

- $m^2 = b_1 t + b_2 t^2 + \dots$:

$$\chi/\beta \sim t^{-\gamma}\left(1 + d_1 t^{\theta} + d_2 t^{2\theta} + \dots + e_1 t + e_2 t^2 + \dots\right)$$

where

$$\gamma = 1 \quad \text{and} \quad \theta = \frac{d-4}{2}$$

The susceptibility

- Expression of the susceptibility in $d = 5, 6, 7$ and 8 :

$$\begin{aligned}
\chi(T)/\beta &= \Gamma t^{-1} + B t^{-\frac{1}{2}} + C + D t^{\frac{1}{2}} + \dots\,, \quad \text{for } d = 5,\\
\chi(T)/\beta &= \Gamma t^{-1} + B \ln t + C + D t \ln t + \dots\,, \quad \text{for } d = 6,\\
\chi(T)/\beta &= \Gamma t^{-1} + C + D t^{\frac{1}{2}} + \dots\,, \quad \text{for } d = 7,\\
\chi(T)/\beta &= \Gamma t^{-1} + C + D t + \dots\,, \quad \text{for } d = 8.
\end{aligned}$$

- for $d > 6$, $\theta = \frac{d-4}{2} > 1$: the analytic terms overhelm the non-analytic one
- $6d$: logarithmic corrections

The susceptibility

- t and τ can be exchanged :

$$
\begin{aligned}
t &= \frac{\tau}{1-\tau} = \tau + \tau^2 + \tau^3 + \dots \\
\tau &= \frac{t}{1+t} = t - t^2 + t^3 - \dots
\end{aligned}
$$

The susceptibility

$$\chi(T)/\beta = \Gamma\tau^{-1} + B\tau^{-1/2} + C(\tau)\,, \quad \text{for } d = 5,$$

$$\chi(T)/\beta = \Gamma\tau^{-1} + B\ln(\tau) + C(\tau)\,, \quad \text{for } d = 6,$$

$$\chi(T)/\beta = \Gamma\tau^{-1} + C(\tau)\,, \quad \text{for } d = 7 \text{ and } d = 8,$$

where

$$\begin{array}{lll} \lim_{\tau\to 1} & C(\tau) = C & \text{and} \\ C & = 1 - \Gamma - B & \text{for } d = 5 \\ C & = 1 - \Gamma & \text{for } d = 6, 7 \text{ and } 8 \end{array}$$

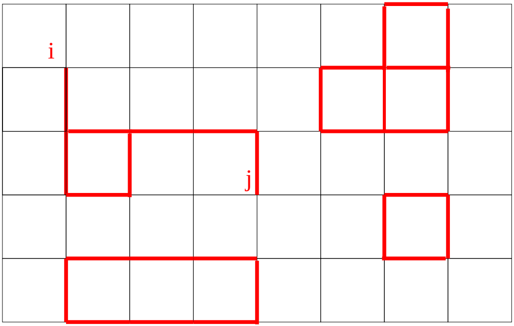
The Worm Algorithm

- Worm Algorithm :
  Prokof'ev, Svistunov and Tupitsyn (*Phys. Lett. A* 1998)
- Application to MCHTSE :
  Prokof'ev and Svistunov (*PRL* 2001)
- Method based on High-Temp. Series Expansion of $\langle s_i s_j \rangle$:

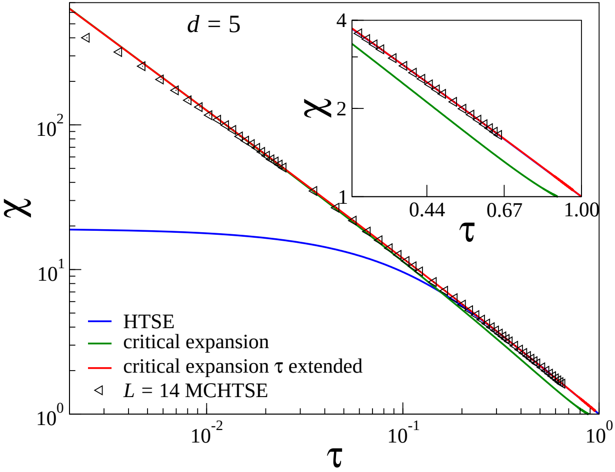
$$\begin{aligned} \langle s_i s_j \rangle &= \frac{1}{Z} \sum_{s} s_i s_j e^{-\beta \sum_{k,l} s_k s_l} \\ &= \frac{1}{Z} \cosh(\beta)^{dN} \sum_{s} s_i s_j \prod_{k,l} (1 + s_k s_l \tanh(\beta)) \end{aligned}$$

The Worm Algorithm

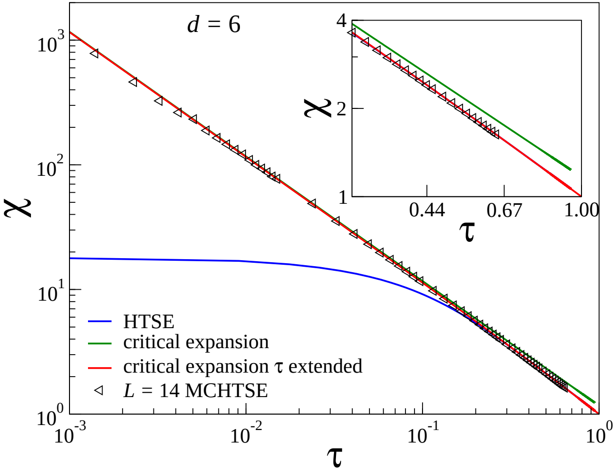
- Sites i and j are called sources.



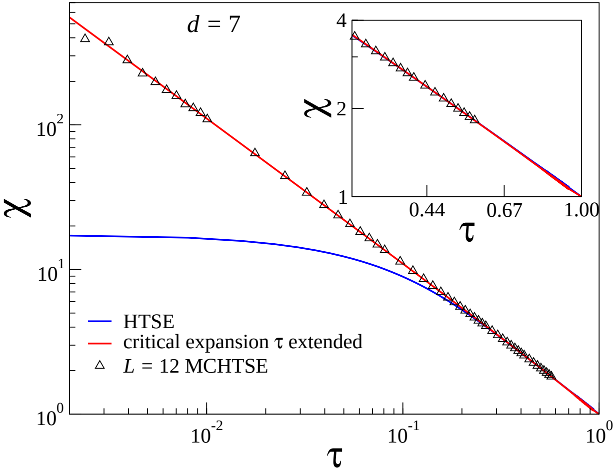
d=5


d = 5
χ
τ
10^2
10^1
10^0
10^-2
10^-1
10^0
4
2
1
0.44
0.67
1.00
HTSE
critical expansion
critical expansion τ extended
L = 14 MCHTSE

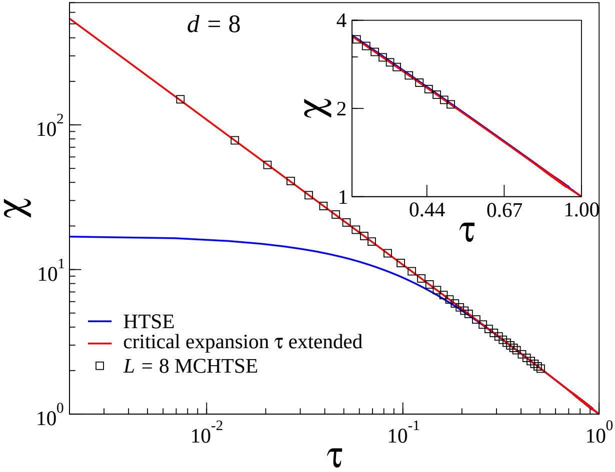
d=6


d = 6
10^3
10^2
10^1
10^0
10^-3
10^-2
10^-1
10^0
χ
τ
4
2
1
0.44
0.67
1.00
χ
τ
HTSE
critical expansion
critical expansion τ extended
L = 14 MCHTSE

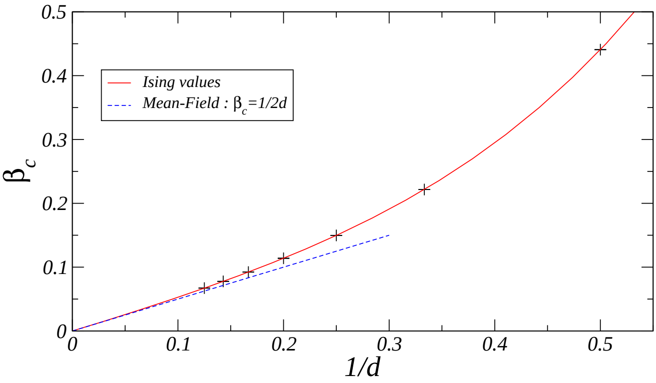
d=7


d = 7
HTSE
critical expansion τ extended
L = 12 MCHTSE
χ
τ
10^2
10^1
10^0
10^-2
10^-1
10^0
4
2
1
0.44
0.67
1.00

d=8


d = 8
χ
τ
10^2
10^1
10^0
10^-2
10^-1
10^0
4
2
1
0.44
0.67
1.00
χ
τ
HTSE
critical expansion τ extended
L = 8 MCHTSE

β_c versus $1/d$



Summary

- Improvement of the estimations of T_c and of the amplitudes of corrective terms.
- Numerical proof of logarithmic correction in $d = 6$.
- Extended scaling (susceptibility) valid :
  - in $d = 5$ and $d = 6$ with non-analytic leading corrections
  - in $d = 7$ and $d = 8$ with analytic leading corrections
- Work available at :
  J. Stat. Mech. (2008) P11010
  ArXiv : cond-mat/0807.2546