Information Geometry and Phase Transitions

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Plan of Talk

- Some general background
- Statistical mechanics
- Some puzzles - black holes *et.al.*

Closeness of distributions

- Given a distribution: $p(x|\theta)$

$p(x|\theta)$

- And a sample: $x_1, \ldots, x_n$
- Estimate a parameter: θ
- Use log likelihood: $\ln L(\theta) = \sum_{i=1}^{n} \ln p(x_i|\theta)$

Closeness of distributions II

- Derivative is score: $$U(\theta) = \frac{d \ln L(\theta)}{d\theta}$$
- Variance is: $$\mathrm{Var}[U(\theta)] = \mathrm{E}\left[\frac{-d^2 \ln L(\theta)}{d\theta^2}\right]$$

Closeness of distributions III

$$\ln L(\theta + \delta\theta) - \ln L(\theta) = \delta\theta \left.\frac{d \ln L(\theta)}{d\theta}\right|_\theta + \frac{(\delta\theta)^2}{2} \left.\frac{d^2 \ln L(\theta)}{d\theta^2}\right|_\theta$$

Fisher information

$$G_{ij}(\theta) = -E\left[\frac{\partial^2 \ln p(x|\theta)}{\partial\theta_i \partial\theta_j}\right] = -\int p(x|\theta) \frac{\partial^2 \ln p(x|\theta)}{\partial\theta_i \partial\theta_j} dx$$

An example: Gaussian

$$p(x|\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

$$G_{ij} = \frac{1}{\sigma^2}\left(\begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array}\right)$$

$$R = -1$$

Statistical Mechanics

$$p(x|\theta) = \exp(-\theta_1 H_1 - \theta_2 H_2 - \ln Z)$$

$$G_{ij} = \partial_i \partial_j \ln Z$$

$$\mathcal{R} = -\frac{1}{2G^2} \begin{vmatrix} \partial_\beta^2 f & \partial_\beta \partial_h f & \partial_h^2 f \\ \partial_\beta^3 f & \partial_\beta^2 \partial_h f & \partial_\beta \partial_h^2 f \\ \partial_\beta^2 \partial_h f & \partial_\beta \partial_h^2 f & \partial_h^3 f \end{vmatrix} ,$$

Statistical Mechanics: ideal gas

$$p(H, V|\alpha, \beta) = \frac{\exp(-\beta H - \alpha V)}{Z(\alpha, \beta)}$$

$$Z(\alpha, \beta) = \left(\frac{2\pi m}{h^2 \beta}\right)^{3N/2} \alpha^{-(N+1)}$$

$$G_{ij} = \begin{pmatrix} \alpha^{-2} & 0 \\ 0 & \frac{3}{2}\beta^{-2} \end{pmatrix}$$

$$R = 0$$

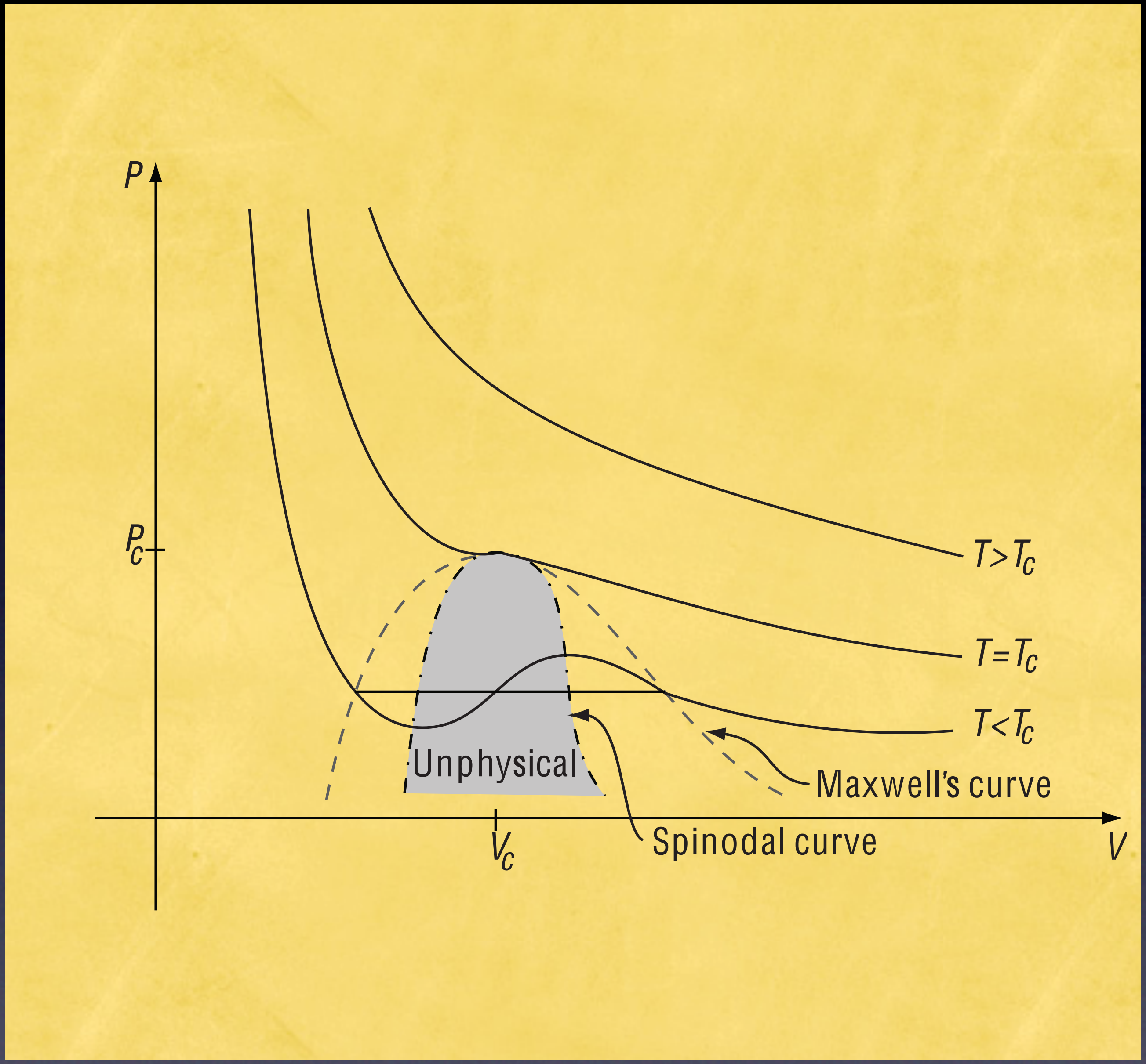
Statistical Mechanics:vdW

$$\left(P + a\frac{N^2}{V^2}\right)(V - bN) = Nk_BT$$

$$R = \frac{4}{3D^2}\left(\frac{\alpha\beta}{\bar{v}}\right)\left(\frac{\alpha\beta}{\bar{v}^3} - D\right)$$

$$D(\alpha, \beta) = \frac{2\alpha\beta}{\bar{v}^3} - \frac{1}{(\bar{v} - b)^2}$$

Statistical Mechanics:vdW



Examples

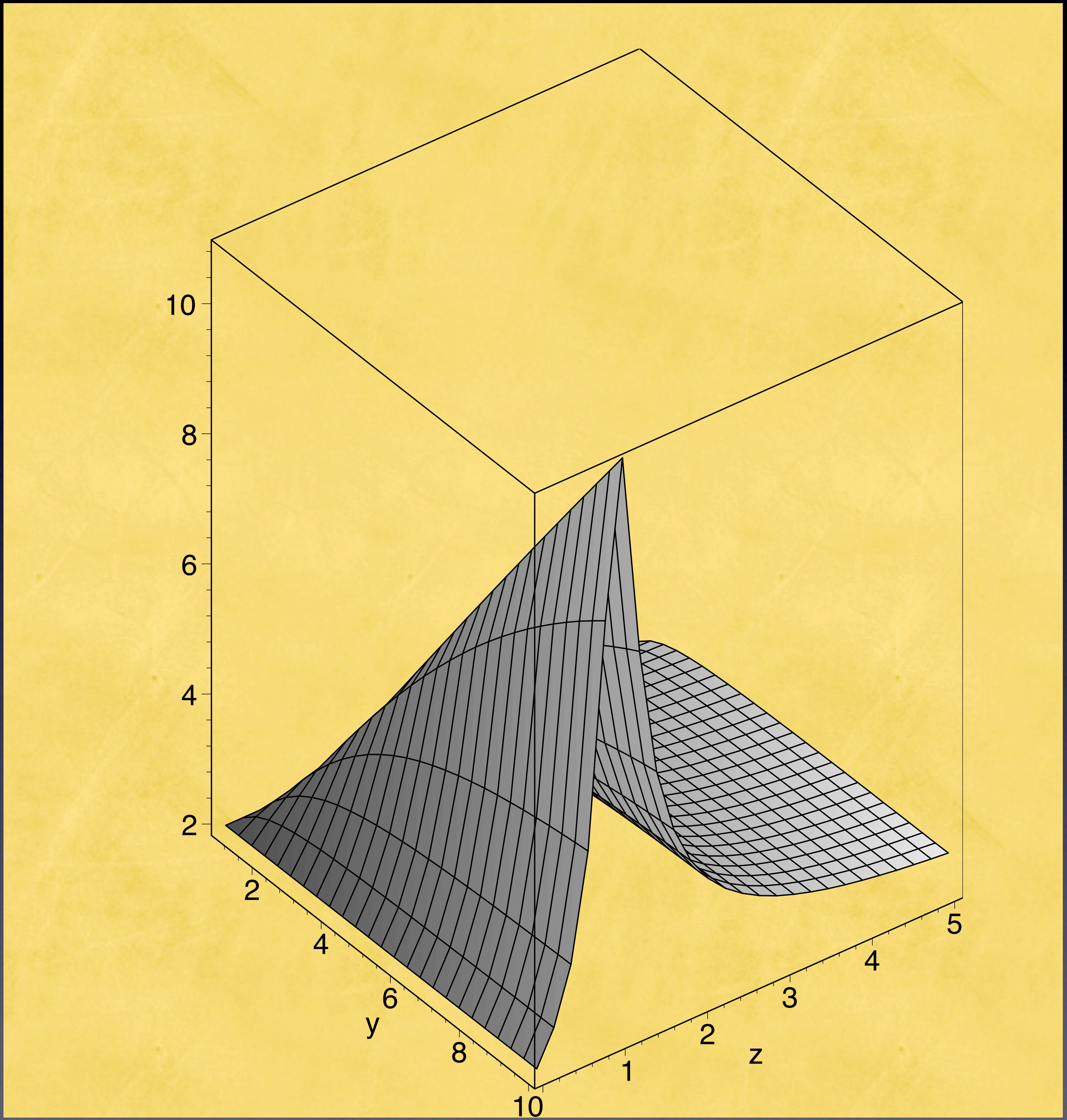
1D Ising (and Potts)

$$\mathcal{R}_{\mathrm{Ising}} = 1 + \frac{\cosh h}{\sqrt{\sinh^2 h + e^{-4\beta}}}$$

Ising + 2D gravity

$$\epsilon = \exp(-\beta) \qquad \mathcal{R} \;=\; -\frac{1}{2G^2} \begin{vmatrix} \frac{352}{225} & 0 & \frac{3}{20}\epsilon_u^{-2} \\ -\frac{1072}{675} & 0 & \frac{3}{20}\epsilon_u^{-3} \\ 0 & \frac{3}{20}\epsilon_u^{-3} & 0 \end{vmatrix} .$$

1D Ising curvature



What's so special about f ?

Nothing - we can Legendre transform

$$G_{ij} = -\partial_i \partial_j S$$ Ruppeiner

$$G_{ij} = \partial_i \partial_j E$$ Weinhold

Black hole stat mech

Consistent results *except* for black holes

Kerr-Newman $4d$

$$dM = TdS + \Omega_H dJ + \phi dQ$$

$$M = \left[\frac{\pi J^2}{S} + \frac{S}{4\pi}\left(1 + \frac{\pi Q^2}{S}\right)^2\right]^{1/2}$$

Inconsistent curvatures

$J = 0$ Reissner-Nordstrom

$Q = 0$ Kerr

Different results from different thermodynamic potential

Inconsistent curvatures

Kerr Higher d

$$M = \frac{d-2}{4} S^{\frac{d-3}{d-2}} \left(1 + \frac{4J^2}{S^2}\right)^{\frac{1}{d-2}}$$

Weinhold metric is flat

$$R = -\frac{1}{S} \frac{1 - 12\frac{d-5}{d-3}\frac{J^2}{S^2}}{\left(1 - 4\frac{d-5}{d-3}\frac{J^2}{S^2}\right)\left(1 + 4\frac{d-5}{d-3}\frac{J^2}{S^2}\right)}$$

Ruppeiner ain't

Quevedo's suggestion

$$G = (d\Phi - \delta_{ab} I^a dE^b)^2 + (\delta_{ab} E^a I^b)(\delta_{cd} dE^c dI^d)$$

$$\{\Phi, E^a, I^a\} \rightarrow \{\tilde{\Phi}, \tilde{E}^a, \tilde{I}^a\}$$

$$\Phi = \tilde{\Phi} - \delta_{kl} \tilde{E}^k \tilde{I}^l \qquad E^i = -\tilde{I}^i \qquad E^j = \tilde{E}^j$$

$$I^i = \tilde{E}^i \qquad I^j = \tilde{I}^j$$

Quevedo’s suggestion

$$R_{RN} = -\frac{8\pi^2 Q^2 S^2 (\pi Q^2 - 3S)}{(\pi Q^2 + S)^3 (\pi Q^2 - S)^2}$$

$S = \pi Q^2$ $\qquad$ $M = Q$ $\qquad$ extremal

$S = \dfrac{\pi Q^2}{3}$ $\qquad$ $M = \dfrac{2Q}{\sqrt{3}}$ $\qquad$ flat

Conclusions

- Information geometry can be applied to stat mech
- Critical behaviour (and scaling) accessible from curvature
- Black hole thermodynamics puzzling
- Legendre invariance?