PASEP and continued fractions

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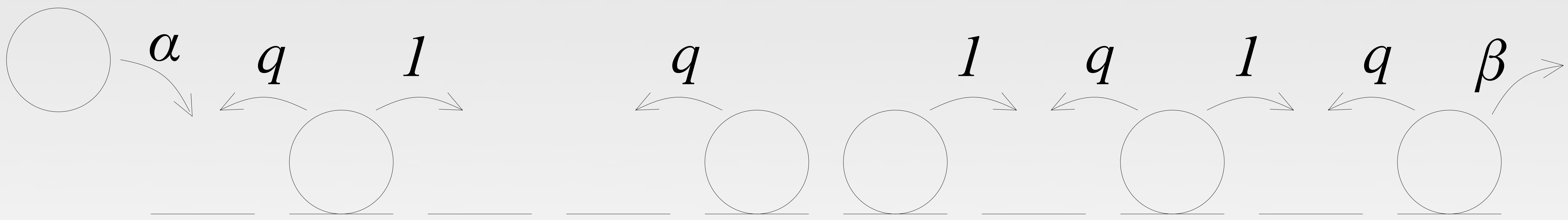
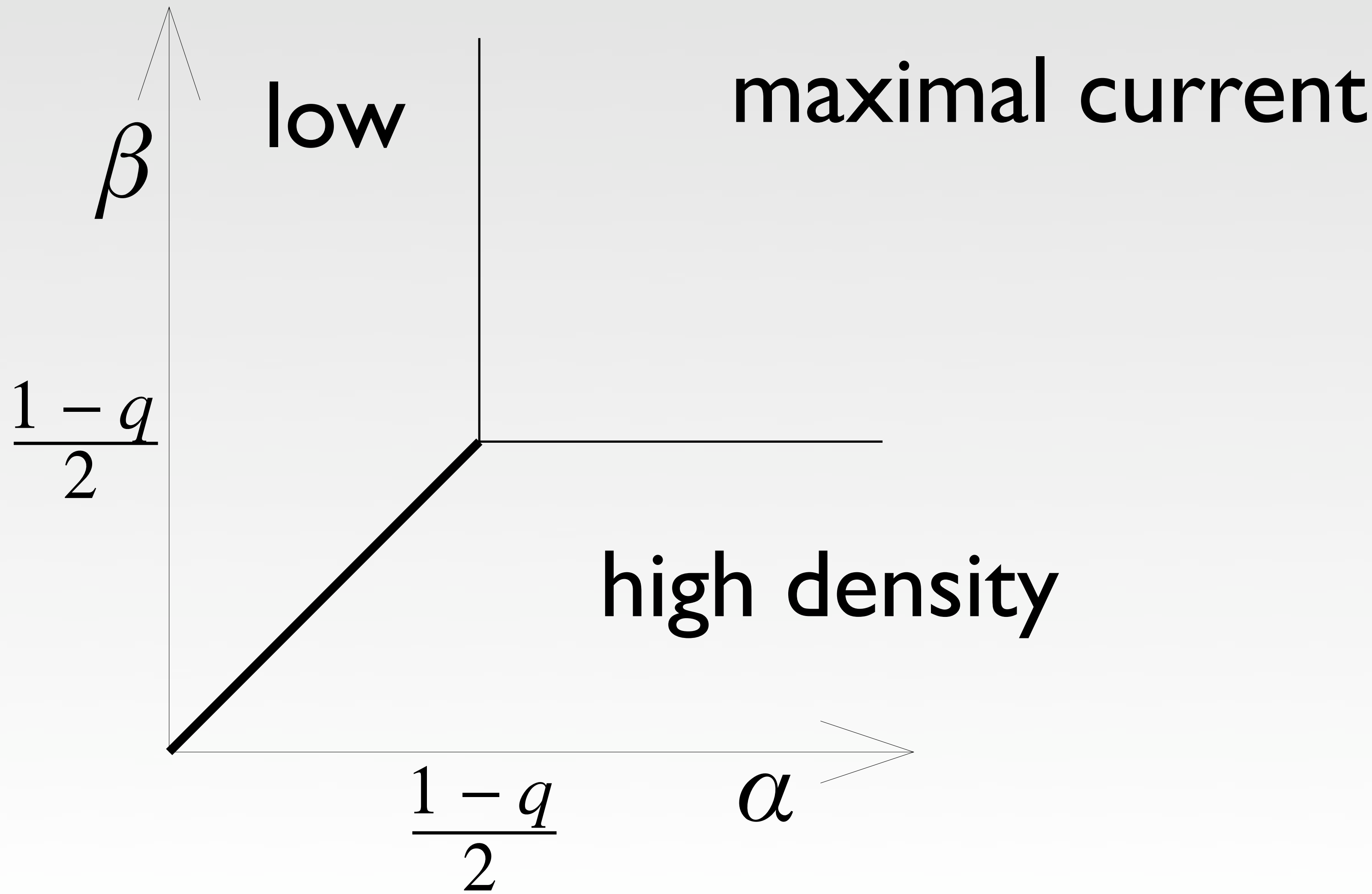
Dyck paths, Motzkin paths and traffic jams
J. Stat. Mech. (2004) P10007
The grand-canonical asymmetric exclusion process and the one-transit walk
J. Stat. Mech. (2004) P06001

- PASEP

- Allow backward jumps



- Phase Diagram



- C configurations
- f(C) Un-normalized probability
- Z Normalization
- W Transition rate

$$Z = \sum f(C)$$

$$\sum_{C' \neq C} [f(C')W(C' \to C) - f(C)W(C \to C')] = 0$$

- Use a string of "words" to enumerate configurations
- Could think of these as matrices
- Sandwich between vectors

$$P = \langle W|DEDDDDEEDDEE|V\rangle$$

$$\langle W|E = \frac{1}{\alpha}\langle W|$$

$$D\,|V\rangle = \frac{1}{\beta}\,|V\rangle$$

$$C = D + E$$

$$Z_L = \langle W|C^L|V\rangle$$

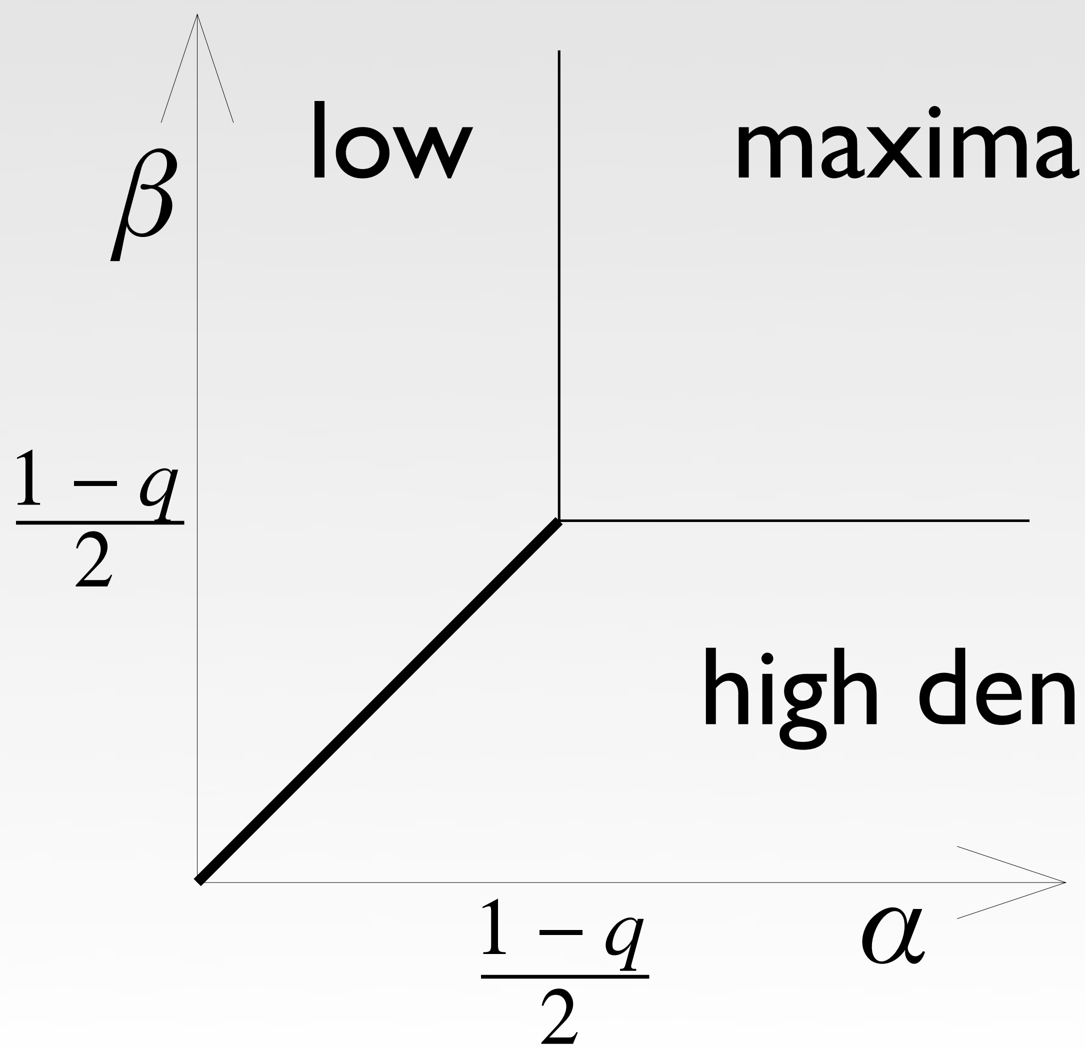
- Representation of PASEP algebra

$$DE - qED = D + E$$

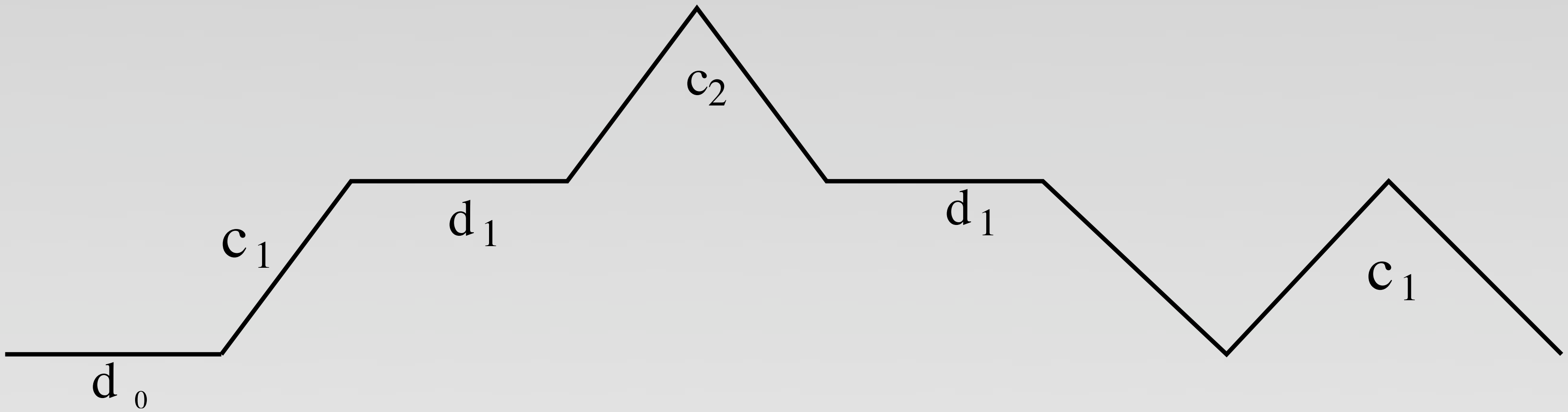
$$C_q = D_q + E_q = \frac{1}{1-q}\begin{pmatrix} 2+a+b & \sqrt{c_1} & 0 & \cdots \\ \sqrt{c_1} & 2+(a+b)q & \sqrt{c_2} & \\ 0 & \sqrt{c_2} & 2+(a+b)q^2 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

$$a = \frac{1-q}{\alpha} - 1 \qquad b = \frac{1-q}{\beta} - 1$$

- Phase Diagram (again!)



Take C as transfer matrix



$$\mathcal{Z}(a,b,q,z) = \cfrac{1}{1 - d_0 z - \cfrac{c_1 z^2}{1 - d_1 z - \cfrac{c_2 z^2}{1 - d_2 z - \cfrac{c_3 z^2}{\cdots}}}}$$

$$d_n = \frac{2 + (a+b)q^n}{1-q} \qquad c_n = \frac{(1-q^n)(1-abq^{n-1})}{(1-q)^2}$$

- Singular behaviour

$$\mathcal{Z}(a,b,q,z) = \cfrac{1}{1 - d_0 z - \cfrac{c_1 z^2}{1 - d_1 z - \cfrac{c_2 z^2}{1 - d_2 z - \cfrac{c_3 z^2}{\cdots}}}}$$

so $$\mathcal{Z}(a,1/a,q,z) \simeq \frac{1}{1 - d_0 z}$$

$$z_{cr} \simeq \frac{1}{d_0} = \frac{1-q}{2+a+1/a}$$

- Maximal current - take into account whole fraction

$$\mathcal{Z}(a,b,q,z) = \cfrac{1}{1 - d_0 z - \cfrac{c_1 z^2}{1 - d_1 z - \cfrac{c_2 z^2}{1 - d_2 z - \cfrac{c_3 z^2}{\dots}}}}$$

Worpitzsky's Theorem

$$\frac{4c_n z_{cr}^2}{(1 - d_{n-1} z_{cr})(1 - d_n z_{cr})} = 1 \ , \ \forall n$$

giving

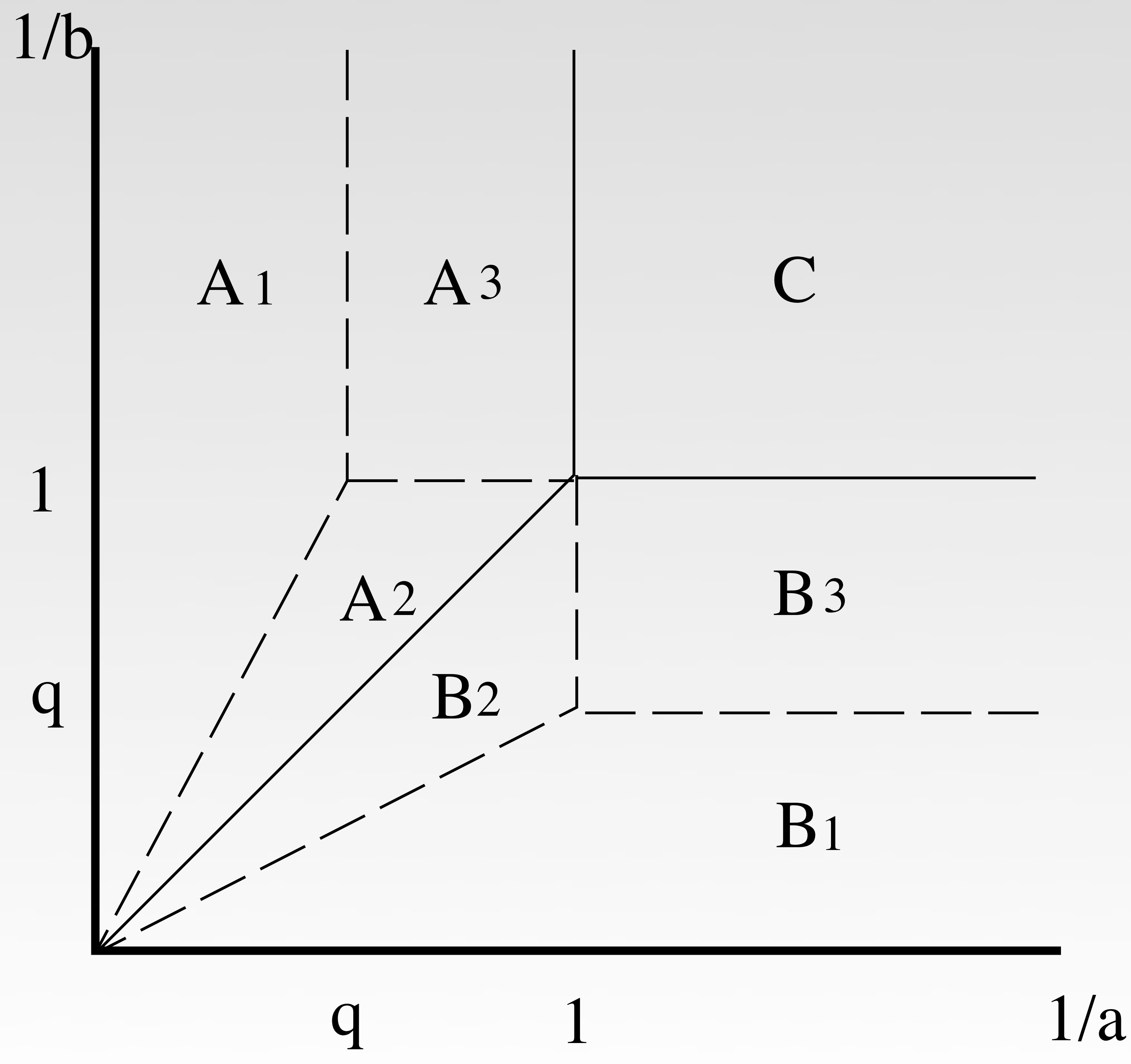
$$z_{cr} \to \frac{1-q}{4} \quad , \quad n \to \infty$$

- Correlation functions *two* singularities

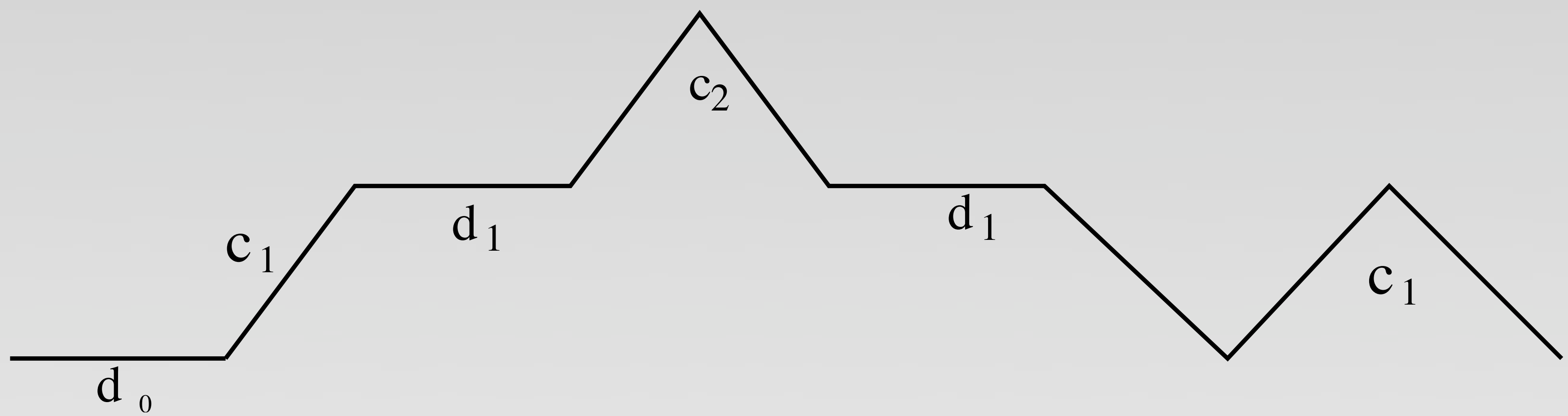
$$\mathcal{Z}_2(a,b,q,z) = \frac{1}{1 - d_0 z - \dfrac{c_1 z^2}{1 - d_1 z}}$$

$$= \frac{1 - d_1 z}{1 - (d_0 + d_1) z + (d_0 d_1 - c_1) z^2}$$

Pick up various sub-phases



Still makes sense for $q > 1$



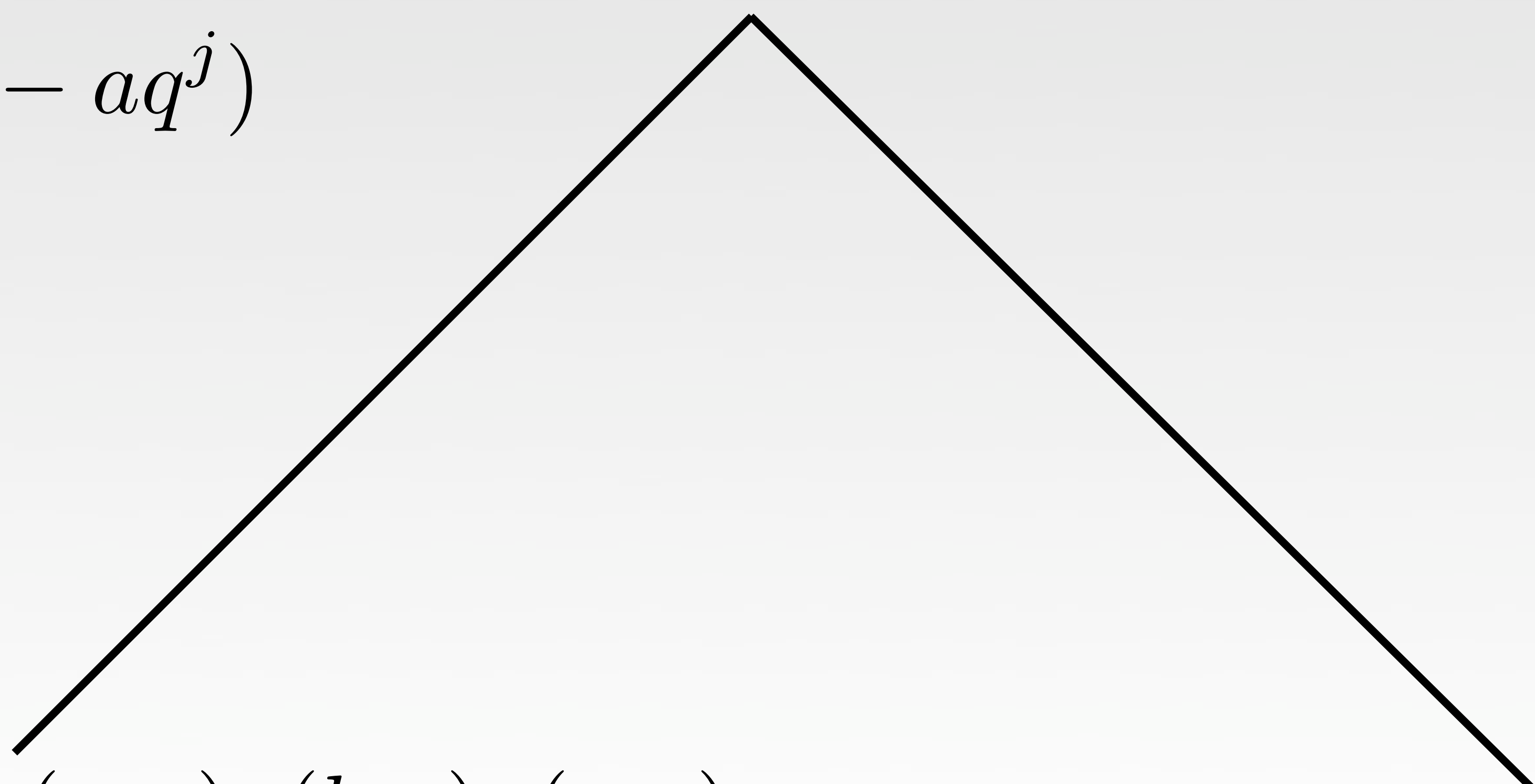
$$\mathcal{Z}(a,b,q,z) = \cfrac{1}{1 - d_0 z - \cfrac{c_1 z^2}{1 - d_1 z - \cfrac{c_2 z^2}{1 - d_2 z - \cfrac{c_3 z^2}{\cdots}}}}$$

$$d_n = \frac{2 + (a+b)q^n}{1-q} \qquad c_n = \frac{(1-q^n)(1-abq^{n-1})}{(1-q)^2}$$

- Dominant behaviour is now

$$Z_N \sim (q^{-1}ab, 1/ab; q^{-1})_\infty \left(\frac{\sqrt{ab}}{q-1}\right)^N q^{\frac{1}{4}N^2}$$

$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$$



$$(a,b,c,\ldots;q)_n = (a;q)_n(b;q)_n(c;q)_n \ldots \; .$$

- Path representation useful for studying (P)ASEP
- Continued Fractions are useful for studying the paths
- Phase transition in (P)ASEP when $q<1$ correspond to (un)binding transitions in the paths
- $q>1$ “inflation”