Entanglement of quantum many-body systems

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AGENDA

- Introduction
  - von Neumann entropy
  - Entanglement in pure quantum systems
    - Area-law
    - Critical systems
      - $d = 1$
      - $d > 1$
- Random quantum systems
  - Strong disorder RG approach
  - Quantum spin chains
  - Quantum magnets in $d = 2$
- Conclusion

Entanglement - von Neumann entropy

Quantum system with a partition $\mathbf{A}$ and $\mathbf{B}$

basis: $|\varphi_{ik}\rangle = (|\varphi_i^{\mathcal{A}}\rangle \otimes |\varphi_k^{\mathcal{B}}\rangle)$

ground state: $|0\rangle = \sum_i^{\mathcal{A}} \sum_k^{\mathcal{B}} c(i,k)(|\varphi_i^{\mathcal{A}}\rangle \otimes |\varphi_k^{\mathcal{B}}\rangle)$

Reduced density matrix: $\rho_A = Tr_B|0\rangle\langle 0|$ and $\rho_B = Tr_A|0\rangle\langle 0|$

matrix-elements:

$$\langle\varphi_i^{\mathcal{A}}|\rho_A|\varphi_j^{\mathcal{A}}\rangle = \rho_A(i,j) = \sum_k^{\mathcal{B}}(\langle\varphi_i^{\mathcal{A}}| \otimes \langle\varphi_k^{\mathcal{B}}|)0\rangle\langle 0(|\varphi_k^{\mathcal{B}}\rangle \otimes |\varphi_j^{\mathcal{A}}\rangle) = \sum_k^{\mathcal{B}} c(i,k)c^*(j,k)\ .$$

von Neumann entropy:

$$S_{A/B} = -\mathrm{Tr}(\rho_A \log_2 \rho_A) = -\mathrm{Tr}(\rho_B \log_2 \rho_B)$$

Infinite d-dimensional system, $\mathbf{A}$ is a hypercube with ℓ^d sites

(or the full system is of L^d sites and $\ell/L = O(1) = const.$)

Noncritical systems

$$S_\ell \sim \ell^{d-1}\ \boxed{\text{area-law}}$$

Critical systems

$\boxed{d = 1}$

$$S_\ell = \frac{c}{3} \log_2 \ell + c_1$$

c - central charge of the associated conformal field theory

(Calabrese & Cardy 2004)

$c < 1$ unitary, minimal models

$$c = 1 - \frac{6}{m(m+1)}$$

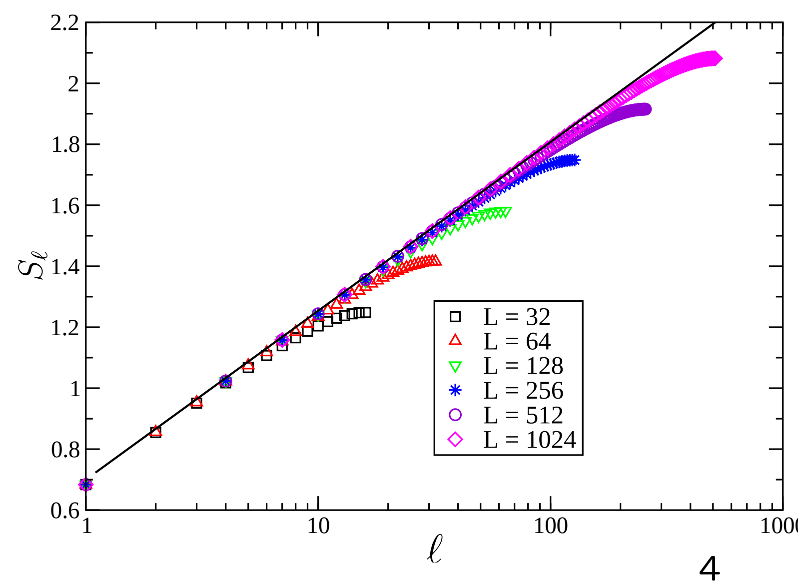
- m=3, c=1/2, Ising model

$$\boxed{H_I = -J \sum_i \sigma_i^x \sigma_{i+1}^x - h \sum_i \sigma_i^z}$$

$\sigma^{x,z}$ Pauli matrices, cr. point: $J = h$

- m=4, c=7/10, tricritical Ising model

- m=5, c=4/5, 3-state Potts model



$m = \infty$, c=1, XXZ spin chain

$$\boxed{H_{XXZ} = J \sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z)}$$

S_i^α: spin-1/2 operators

For finite systems:

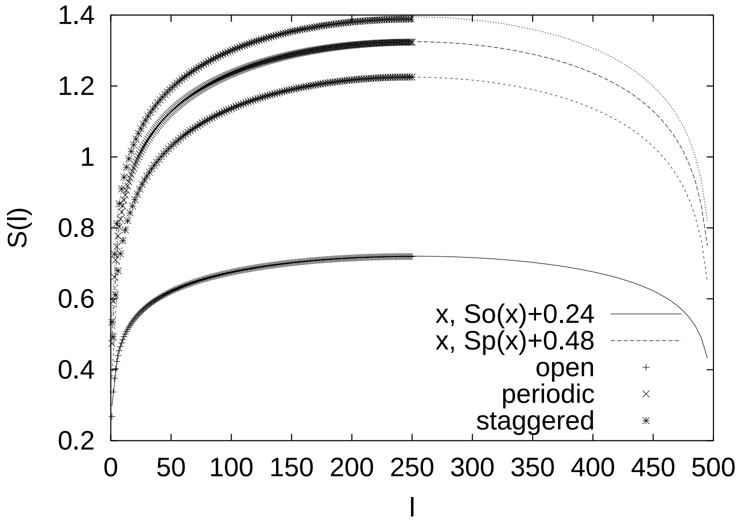
periodic:

$$S_{\mathcal{A}} = \frac{c}{3} \log_2 \left[\frac{L}{\pi} \sin\left(\frac{\ell\pi}{L}\right)\right] + c_1$$

open:

$$S_{\mathcal{A}} = \frac{c}{6} \log_2 \left[\frac{2L}{\pi} \sin\left(\frac{\ell\pi}{L}\right)\right] + \log_2 g + c_1/2$$

$\log_2 g$ is the surface entropy



In the vicinity of the critical point:

$$S_{\mathcal{A}} = \frac{c}{3} \log_2 \xi + c_1$$

$\ell \to \xi$ is the (entanglement) correlation length

.

At finite temperature:

$$S_{\mathcal{A}}(\beta) = \frac{c}{3} \log_2 \left[\frac{\beta}{\pi} \sinh\left(\frac{\ell\pi}{\beta}\right)\right] + c_1$$

$L \to i\beta$ inverse temperature

$\boxed{d = 2}$

- gapless fermionic systems
  logarithmic correction: $S_\ell \sim \ell^{d-1} \log_2 \ell$

- gapless bosonic systems
  no logarithmic correction: $S_\ell \sim \ell^{d-1}$

Random quantum systems

Random transverse-field Ising model

$$\boxed{H_I = -\sum_{i,j} J_{ij}\sigma_i^x\sigma_j^x - \sum_i h_i\sigma_i^z}$$

$\boxed{d=1}$ critical point from exact results:

$$\boxed{\delta = \overline{\ln J} - \overline{\ln h} = 0}$$

$\boxed{d=2}$ critical point from numerical results

uniform distribution for J_i:

$$P(J) = \frac{1}{\Delta}, \quad \text{for} \quad 1 - \frac{\Delta}{2} \leq \mathrm{J} \leq 1 + \frac{\Delta}{2}$$

,

constant transverse field: $h_i = h_0$

critical point: $h_0 = h_c$

Strong disorder RG approach

(Ma & Dasgupta 1979, D.S. Fisher 1992, F.I. & Monthus 2005)

- sort the couplings and transverse fields, $\Omega = \max(J_i, h_i)$
- eliminate the largest parameter - reduce the number of spins by one
- generate new effective parameters between the remaining spins
  - $\boxed{\Omega = J_{ij}}$ i and j form a ferromagnetic cluster

    in an effective field: $\tilde{h}_{(ij)} = \frac{h_i h_j}{J_{ij}}$
  - $\boxed{\Omega = h_i}$ site i is decimated out

    new effective couplings between sites j and k: $\tilde{J}_{jk} = \frac{J_{ij} J_{ik}}{h_i}$
- repeate the transformation
- at the fixed point Ω is reduced to $\Omega^* = 0$.

Infinite disorder fixed point

- Distribution of the effective parameters is logarithmically broad

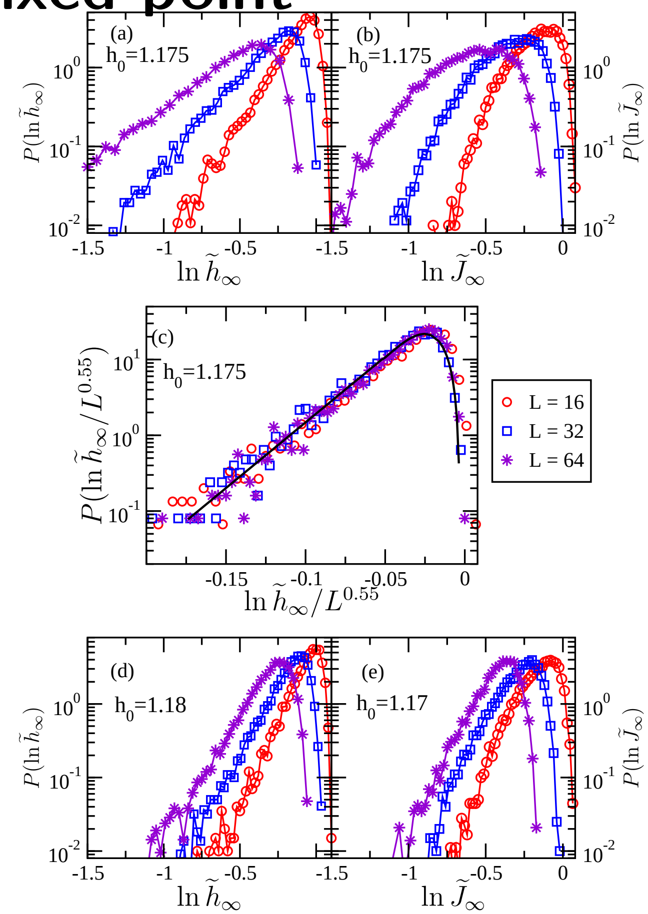
  Asymptotically exact decimation steps

- Strongly anisotropic scaling $\ln \Omega_L \sim L^{\psi}$

  $d = 1$: $\psi = 0.5$

  $d = 2$: $\psi = 0.55$

- In the Griffiths phase: $\Omega \sim L^{-z}$



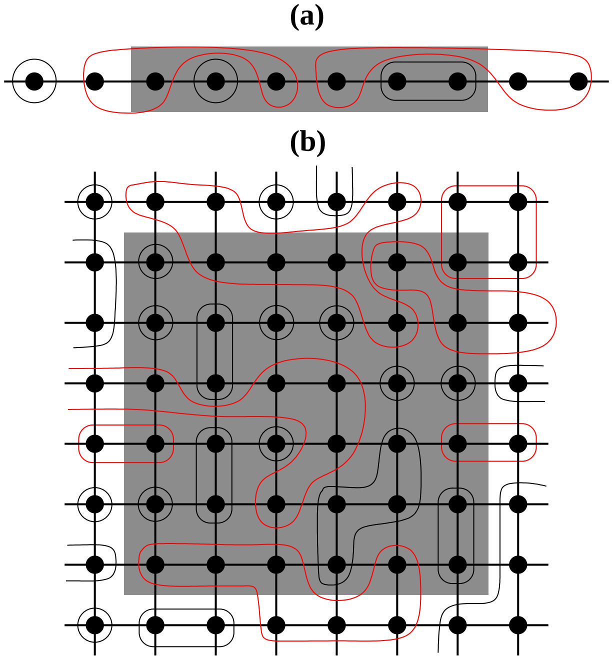
Structure of the ground state

- collection of independent ferromagnetic clusters

- each cluster of n spins is frozen in an entangled state

$$\frac{1}{\sqrt{2}}(|\underbrace{\uparrow\uparrow\cdots\uparrow}_{n\ \text{times}}\rangle + |\underbrace{\downarrow\downarrow\cdots\downarrow}_{n\ \text{times}}\rangle)$$

- entanglement of a block: number of clusters that connect sites inside to sites outside the block



Estimate of the entropy

1d

- r: length scale $\rightarrow$ average size of clusters
- n_r: fraction of *active* (undecimated) spins $\rightarrow$ $n_r \sim 1/r$
- *active* spins form clusters across the boundary of the block with finite probability
- sum of contributions for $r < \ell$: $\overline{S}_\ell \sim \int_{r_0}^{\ell} \mathrm{d}r\, n_r \sim \boxed{\ln \ell}$
- exact evaluation (Refael & Moore, 2004): $\overline{S}_\ell = \frac{\ln 2}{6} \log_2 \ell$

2d

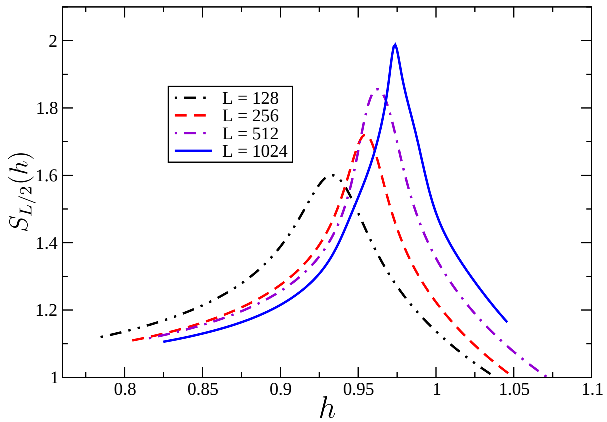
- r: length scale $\rightarrow$ average size of clusters
- n_r: fraction of *active* spins at the surface $\rightarrow$ $n_r \sim \ell/r$
- from the *active* spins there are $\sim \ln r$ which are engaged in the surface layer
- from the *active* spins $O(1)$ form clusters across the boundary of the block
- sum of contributions for $r < \ell$: $\overline{S}_\ell \sim \int_{r_0}^{\ell} \mathrm{d}r\, n_r / lnr \sim \boxed{\ell \ln \ln \ell}$

Application: finite-size transition points in 1d

(F.I., Lin, Rieger, Monthus, Phys. Rev. B**76**, 064421 (2007))

Definition

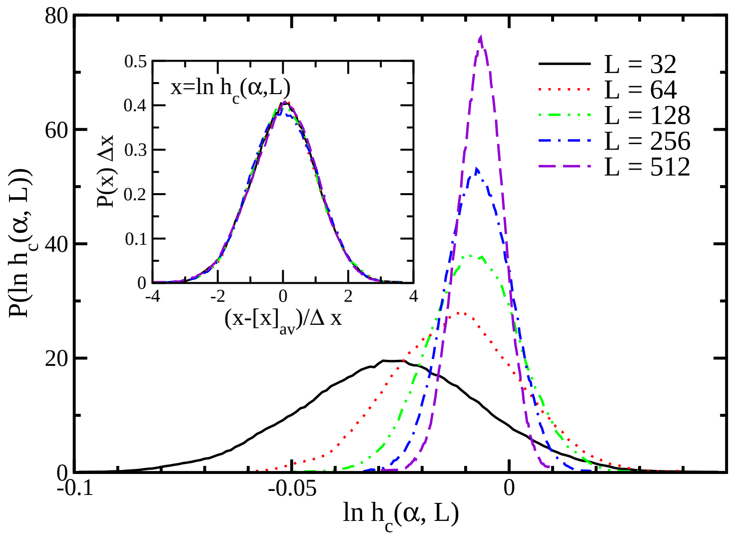
- Traditional definitions does not work
  - *susceptibility* is divergent in the Griffiths phase, too
  - *specific heat* has an essential singularity
- $h_c(\alpha, L)$: position of the maximum of $\overline{S}_L(\alpha, h)$ of a sample, α



calculation by free-fermionic methods (Peschel, 2003)

Distribution of $h_c(\alpha, L)$

- scaling of the shift: $\overline{h_c(\alpha, L)} - h_c(\infty) = \lambda(L) \sim L^{-1/\nu_{typ}}$, $\nu_{typ} = 1$
- scaling of the width: $\Delta h_c(L) \sim L^{-1/\nu}$, $\nu = 2$
- infinite disorder scaling: $\frac{\Delta h_c(L)}{\lambda(L)} \to \infty$



Numerical study in 2d

(Lin, F.I., Rieger, Phys. Rev. Lett. **99**, 147202 (2007))

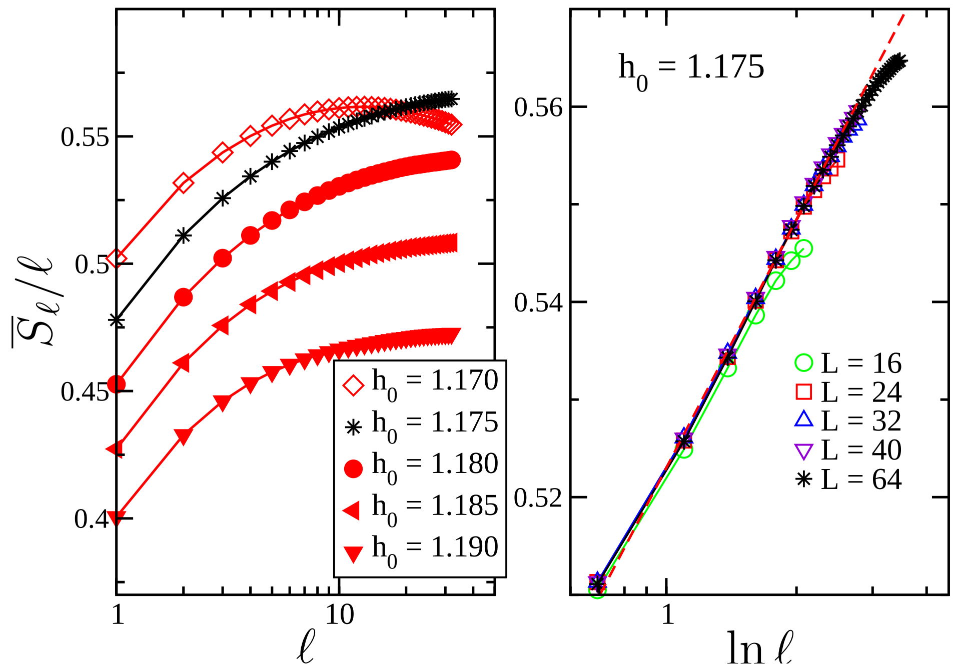
- Calculation of $\overline{S}_\ell(h_0)$ by numerical RG method

- $\overline{S}_\ell(h_0)$ is nondivergent for large ℓ for $h_0 \neq h_c$

- $\overline{S}_\ell(h_0)$ is divergent for large ℓ at $h_0 = h_c$

- at $h_0 = h_c$ we have

$$\overline{S}_\ell \sim \ell \log_2 \log_2 \ell$$



Conclusions

- Random quantum systems are conveniently studied by the SDRG method
- Asymptotically exact results at an infinite disorder fixed point even for $d > 1$
- In the SDRG framework the entanglement entropy has a simple geometrical interpretation
- The entropy/surface is divergent at the critical point of the random transverse-field Ising model
  - d = 1 $\overline{S}_\ell \sim \log_2 \ell$
  - d = 2 $\overline{S}_\ell/\ell \sim \log_2 \log_2 \ell$

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