

# Local scale-invariance in ageing phenomena

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arXiv:0709.3228

CompPhys07, Universität Leipzig, 29<sup>th</sup> of November 2007

# Contents :

- I. Ageing phenomena  
physical ageing ; scaling behaviour and exponents
- II. Hidden dynamical symmetries  
Local scaling with  $z = 2$  ; stochastic field-theory ; computation of response and correlation functions
- III. Local scale-invariance for  $z \neq 2$   
Mass terms ; integrability ; test through responses and correlators in  $2D$  disordered Ising model
- IV. Conclusions

# I. Ageing phenomena

- why do materials 'look old' after some time ?
- which (reversible) microscopic processes lead to such macroscopic effects ?
- **physical ageing** known since historical (or prehistorical) times
- systematic studies first in glassy systems

STRUIK 78

*a priori* behaviour should depend on entire prehistory

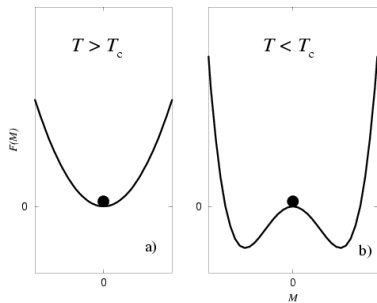
**but** evidence for **reproducible** and **universal** behaviour

- for better conceptual understanding : study ageing first in simpler systems (i.e. disordered ferromagnets)
- **ageing** : defining characteristics and symmetry properties :
  - ① slow dynamics (i.e. non-exponential relaxation)
  - ② breaking of time-translation invariance
  - ③ dynamical scaling
- new evidence for larger, **local scaling** symmetries

easier to study : ageing in simple systems without disorder  
consider a simple magnet (ferromagnet, i.e. Ising model)

- 1 prepare system initially at high temperature  $T \gg T_c > 0$
- 2 **quench** to temperature  $T < T_c$  (or  $T = T_c$ )  
→ non-equilibrium state
- 3 fix  $T$  and observe dynamics

BRAY 94

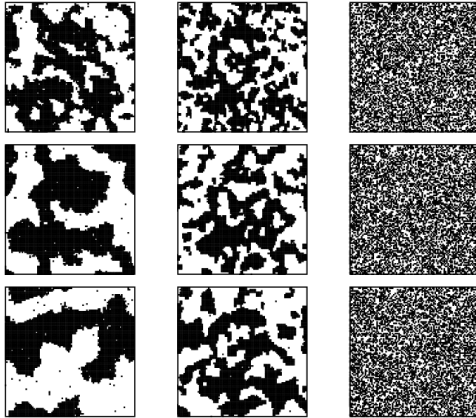


### competition :

at least 2 equivalent ground states  
local fields lead to rapid local ordering  
no global order, relaxation time  $\infty$

formation of ordered domains, of linear size  $L = L(t) \sim t^{1/z}$

**dynamical exponent**  $z$



Snapshots of spin configurations in several  $2D/3D$  Ising models quenched to  $T < T_c$ , for three different times  $t = 25, 100, 225$ .  
**Left** : pure **Middle** : disordered **Right** :  $3D$  spin glass

# Scaling behaviour & exponents

**single** relevant time-dependent length scale  $L(t) \sim t^{1/z}$

BRAY 94, JANSSEN ET AL. 92, CUGLIANDOLO & KURCHAN 90S, GODRÈCHE & LUCK 00, ...

$$\text{correlator } C(t, s; \mathbf{r}) := \langle \phi(t, \mathbf{r}) \phi(s, \mathbf{0}) \rangle = s^{-b} f_C \left( \frac{t}{s}, \frac{\mathbf{r}}{(t-s)^{1/z}} \right)$$

$$\text{response } R(t, s; \mathbf{r}) := \left. \frac{\delta \langle \phi(t, \mathbf{r}) \rangle}{\delta h(s, \mathbf{0})} \right|_{h=0} = s^{-1-a} f_R \left( \frac{t}{s}, \frac{\mathbf{r}}{(t-s)^{1/z}} \right)$$

**No** fluctuation-dissipation theorem :  $R(t, s; \mathbf{r}) \neq T \partial C(t, s; \mathbf{r}) / \partial s$   
values of exponents : equilibrium correlator  $\rightarrow$  classes **S** and **L**

$$C_{\text{eq}}(\mathbf{r}) \sim \begin{cases} \exp(-|\mathbf{r}|/\xi) \\ |\mathbf{r}|^{-(d-2+\eta)} \end{cases} \implies \begin{cases} \text{class } \mathbf{S} \\ \text{class } \mathbf{L} \end{cases} \implies \begin{cases} a = 1/z \\ a = (d-2+\eta)/z \end{cases}$$

if  $T < T_c$  :  $z = 2$  and  $b = 0$       if  $T = T_c$  :  $z = z_c$  and  $b = a$   
for  $y \rightarrow \infty$  :  $f_{C,R}(y, \mathbf{0}) \sim y^{-\lambda_{C,R}/z}$ ,     $\lambda_{C,R}$  independent exponents

Question : **general** arguments to find form of scaling functions?

## II. Hidden dynamical symmetries

Consider possible symmetries of Langevin equations (model A)

HOHENBERG-HALPERIN 77

$$2\mathcal{M} \frac{\partial \phi}{\partial t} = \Delta \phi - \frac{\delta \mathcal{V}[\phi]}{\delta \phi} + \eta$$

non-conserved order-parameter  $\phi(t, \mathbf{r})$ , centred noise  $\eta$  :

$$\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2T \delta(t - t') \delta(\mathbf{r} - \mathbf{r}')$$

fully disordered initial conditions (centred gaussian noise)

**Question** : extended dynamical scaling for given  $z \neq 1$ ? MH 92, 94, 02

**motivation** :

1. conformal invariance in equilibrium critical phenomena,  $z = 1$
2. Schrödinger-invariance of simple diffusion,  $z = 2$

LIE 1881, NIEDERER 72, HAGEN 71, KASTRUP 68

$$t \mapsto \frac{\alpha t}{\gamma t + \delta}, \quad \mathbf{r} \mapsto \frac{\mathcal{R}\mathbf{r} + \mathbf{v}t + \mathbf{a}}{\gamma t + \delta}, \quad \alpha\delta = 1$$

Lie algebra  $\mathfrak{alg}_d \subset \mathfrak{sch}_d$ , projective representations

# Stochastic field-theory

Langevin equations do **not** have non-trivial dynamical symmetries!  
compare results of **deterministic** symmetries to **stochastic** models?  
go to stochastic field-theory, action

JANSSEN, DE DOMINICIS, . . . 70s-80s

$$\mathcal{J}[\phi, \tilde{\phi}] = \underbrace{\int \tilde{\phi}(2\mathcal{M}\partial_t - \Delta)\phi + \tilde{\phi}\mathcal{V}'[\phi]}_{\mathcal{J}_0[\phi, \tilde{\phi}] : \text{deterministic}} - T \underbrace{\int \tilde{\phi}^2 - \int \tilde{\phi}_{t=0} C_{init} \tilde{\phi}_{t=0}}_{+ \mathcal{J}_b[\tilde{\phi}] : \text{noise}}$$

$\tilde{\phi}$  : response field ;

$$C(t, s) = \langle \phi(t)\phi(s) \rangle, R(t, s) = \langle \phi(t)\tilde{\phi}(s) \rangle$$

averages :  $\langle A \rangle_0 := \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} A[\phi, \tilde{\phi}] \exp(-\mathcal{J}_0[\phi, \tilde{\phi}])$

masses :

$$\mathcal{M}_\phi = -\mathcal{M}_{\tilde{\phi}}$$

**Theorem** : **IF**  $\mathcal{J}_0$  is Galilei- and spatially translation-invariant,  
**then** Bargman superselection rules hold true :

$$\left\langle \phi_1 \cdots \phi_n \tilde{\phi}_1 \cdots \tilde{\phi}_m \right\rangle_0 \sim \delta_{n,m} \quad (1)$$

$$\begin{aligned}
 R(t, s) &= \langle \phi(t) \tilde{\phi}(s) \rangle = \langle \phi(t) \tilde{\phi}(s) e^{-\mathcal{J}_b[\tilde{\phi}]} \rangle_0 \\
 &= \langle \phi(t) \tilde{\phi}(s) \rangle_0 = R_0(t, s)
 \end{aligned}$$

Bargman eq. (1)  $\implies$  response function does not depend on noise!

**left side** : computed in stochastic models

**right side** : local scale-symmetry of deterministic equation

**application to ageing** :  $\text{age}_d$ -covariant two-point response function

$$R(t, s; \mathbf{r}) = r_0 s^{-1-a} \left(\frac{t}{s}\right)^{1+a'-\lambda_{R/z}} \left(\frac{t}{s} - 1\right)^{-1-a'} \exp\left(-\frac{\mathcal{M}}{2} \frac{\mathbf{r}^2}{t-s}\right)$$

find  $C(t, s) = \langle \phi(t) \phi(s) \rangle = \langle \phi(t) \phi(s) e^{-\mathcal{J}_b[\tilde{\phi}]} \rangle_0$  from Bargman rule  
 'initial' and 'thermal' contributions; contain **four-point responses**

explicit tests in Ising/Potts models

### III. Local scale-invariance for $z \neq 2$

Extend to  $z \neq 1, 2$  by **generators with mass terms** (for  $d = 1$ ) :

MH 02; BAUMANN & MH 07

$$Y_{1-1/z} := -t\partial_r - \mu zr\nabla_r^{2-z} - \gamma z(2-z)\partial_r\nabla_r^{-z} \quad \text{Galilei}$$

$$X_1 := -t^2\partial_t - \frac{2}{z}tr\partial_r - \frac{2(x+\xi)}{z}t - \mu r^2\nabla_r^{2-z} \quad \text{special} \\ -2\gamma(2-z)r\partial_r\nabla_r^{-z} - \gamma(2-z)(1-z)\nabla_r^{-z}$$

- depend on two parameters  $\gamma, \mu$  and on two dimensions  $x, \xi$
- contains fractional derivative ( $\hat{f}$  : Fourier transform)

$$\nabla_r^\alpha f(\mathbf{r}) := i^\alpha \int_{\mathbb{R}^d} \frac{d\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^\alpha e^{i\mathbf{r}\cdot\mathbf{k}} \hat{f}(\mathbf{k})$$

- some properties :  $\nabla_r^\alpha \nabla_r^\beta = \nabla_r^{\alpha+\beta}$ ,  $[\nabla_r^\alpha, r_i] = \alpha \partial_{r_i} \nabla_r^{\alpha-2}$   
 $\nabla_r^\alpha \exp(i\mathbf{q}\cdot\mathbf{r}) = i^\alpha |\mathbf{q}|^\alpha \exp(i\mathbf{q}\cdot\mathbf{r})$

**Fact 1** : simple algebraic structure :

$$[X_n, X_{n'}] = (n - n')X_{n+n'} \quad , \quad [X_n, Y_m] = \left(\frac{n}{z} - m\right) Y_{n+m}$$

→ Generate  $Y_m$  from  $Y_{-1/z} = -\partial_r$ .

**Fact 2** : LSI-invariant Schrödinger operator :

$$\mathcal{S} := -\mu\partial_t + z^{-2}\nabla_r^2$$

Let  $x_0 + \xi = 1 - 2/z + (2 - z)\gamma/\mu$ . Then  $[\mathcal{S}, Y_m] = 0$  and

$$[\mathcal{S}, X_0] = -\mathcal{S} \quad , \quad [\mathcal{S}, X_1] = -2t\mathcal{S} + \frac{2\mu}{z}(x - x_0)$$

⇒  $\mathcal{S}\phi = 0$  is **lsi-invariant** equation, if  $x_\phi = x_0$ .

**Physical assumption** (hidden) : equations of motion remain of first order in  $\partial_t$ , even after renormalisation.

### Fact 3 : non-trivial conservation laws :

iterated commutator with  $G := Y_{1-1/z}$ ,  $\text{ad}_G \cdot = [\cdot, G]$

$$M_\ell := (\text{ad}_G)^{2\ell+1} Y_{-1/z} = a_\ell \mu^{2\ell+1} \nabla_{\mathbf{r}}^{(2\ell+1)(1-z)+1}$$

For  $z = 2$ ,  $a_\ell = 0$  if  $\ell \geq 1$ . For a  $n$ -point function  $F^{(n)} = \langle \phi_1 \dots \phi_n \rangle$ ,  $M_\ell F^{(n)} = 0$  gives in momentum space

$$\left( \sum_{i=1}^n \mu_i^{2\ell-1} |\mathbf{k}_i|^{2\ell-(2\ell-1)z} \right) \widehat{F}^{(n)}(\{t_i, \mathbf{k}_i\}) = 0$$

$$\left( \sum_{i=1}^n \mathbf{k}_i \right) \widehat{F}^{(n)}(\{t_i, \mathbf{k}_i\}) = 0$$

$\implies$  momentum conservation & conservation of  $|\mathbf{k}|^\alpha$  !

analogous to relativistic factorisable scattering

**Consequence** : a  $\zeta$ -covariant  $2n$ -point function  $F^{(2n)}$  is only non-zero, if the 'masses'  $\mu_i$  can be arranged in pairs  $(\mu_i, \mu_{\sigma(i)})$  with  $i = 1, \dots, n$  such that  $\mu_i = -\mu_{\sigma(i)}$ .

generalized Galilei-invariance with  $z \neq 2 \implies$  'integrability'

**Corollary 1** : Bargman rule :  $\langle \phi_1 \dots \phi_n \tilde{\phi}_1 \dots \tilde{\phi}_m \rangle_0 \sim \delta_{n,m}$

**Corollary 2** : treat (linear) stochastic equations with  $\zeta$ -invariant deterministic part, reduction formulæ

**Corollary 3** : response function noise-independent

$$R(t, s; \mathbf{r}) = R(t, s) \mathcal{F}^{(\mu_1, \gamma_1)}(|\mathbf{r}|(t-s)^{-1/z})$$

$$R(t, s) = r_0 s^{-a} \left(\frac{t}{s}\right)^{1+a'-\lambda_R/z} \left(\frac{t}{s} - 1\right)^{-1-a'}$$

$$\mathcal{F}^{(\mu, \gamma)}(\mathbf{u}) = \int_{\mathbb{R}^d} \frac{d\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^\gamma \exp(i\mathbf{u} \cdot \mathbf{k} - \mu|\mathbf{k}|^z)$$

**Corollary 4** :

Correlators obtained from factorised 4-point responses.

# How to test the foundations of LSI

describes dynamic symmetries of **deterministic part** of Langevin eq.

local scaling theory is built on :

- a) simple scaling – domain sizes  $L(t) \sim t^{1/z}$
- b) invariance under Möbius transformation  $t \mapsto t/(\gamma t + \delta)$
- c) Galilei-invariance generalised to  $z \neq 2$

together with spatial translation-invariance

⇒ extended Bargman rules

⇒ factorisation of  $2n$ -point functions

Möbius transformation	autoresponse $R(t, s)$
generalised Galilei-invariance	space-time response $R(t, s; \mathbf{r})$
factorisation	two-time correlation function

# Tests of LSI for $z \neq 2$ :

- spherical model with conserved order-parameter,  $T = T_c$ ,  
 $z = 4$

BAUMANN & MH 06

- Mullins-Herring model for surface growth,  $z = 4$

RÖTHLEIN, BAUMANN, PLEIMLING 06

- spherical model with long-ranged interactions,  $T \leq T_c$ ,  
 $0 < z = \sigma < 2$

CANNAS ET AL. 01 ; BAUMANN, DUTTA, MH 07

- **2D Ising model with disorder**,  $T < T_c$  (non-frustrated)

Hamiltonian  $\mathcal{H} = - \sum_{i,j} J_{ij} \sigma_i \sigma_j$

**uniform** disorder  $J_{ij} \in [1 - \epsilon/2, 1 + \epsilon/2] \implies T_c(\epsilon) \approx T_c(0)$

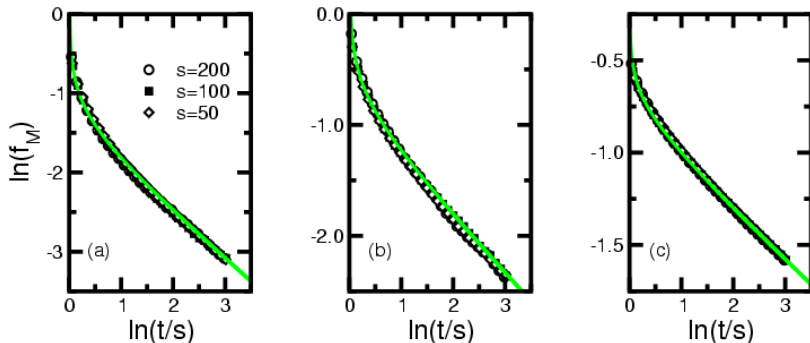
disorder defects 'pin' domain walls  $\implies$  thermal activation

**if** logarithmic barrier heights  $\implies z = 2 + \epsilon/T$ .

PAUL, PURI & RIEGER 04

**change contrôle parameters to vary  $z$**

# Practical tests of LSI, I : autoreponse



Thermoremanent susceptibility

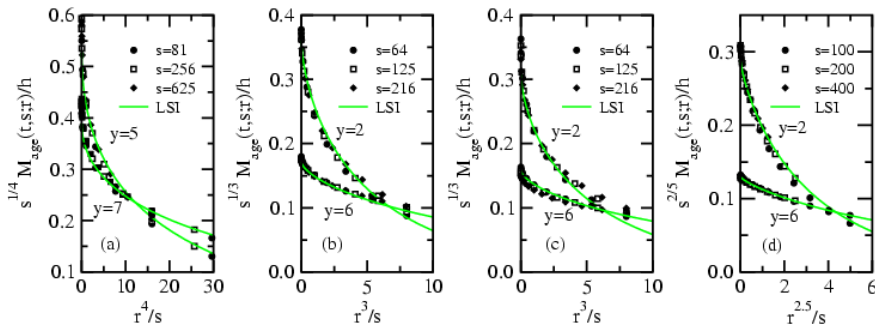
$$\chi_{\text{TRM}}(t, s) = \int_0^s du R(t, u) = s^{-a} f_M(t/s) + O(s^{-\lambda_R/z})$$

Full **curve** : LSI-prediction, with  $a = a' = 1/z$ .

Confirm  $z = 2 + \epsilon/T$  : agreement with PAUL, PURI & RIEGER 04.

MH & PLEIMLING, Europhys. Lett. **76**, 561 (2006).

# Practical tests of LSI, II : space-time response



(a)  $\epsilon = 2, T = 1$  (b)  $\epsilon = T = 1$  (c)  $\epsilon = T = 0.5$  (d)  $\epsilon = 0.5, T = 1$

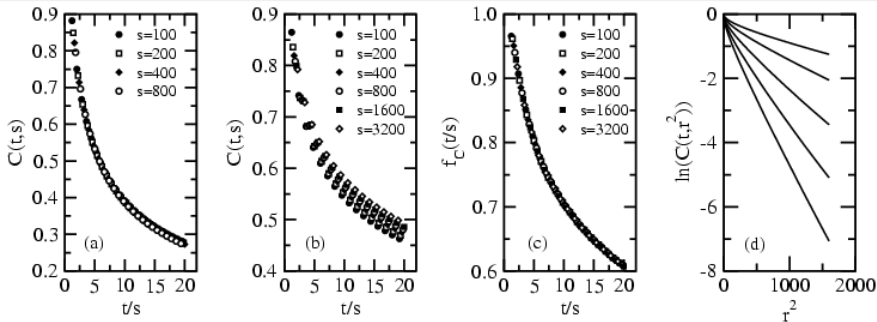
$$\chi_{\text{TRM}}(t, s; \mathbf{r}) = \int_0^s du R(t, u; \mathbf{r}) = s^{-a} r_0 f_M(t/s, \mathbf{r} s^{-1/z}) + O(t^{-\lambda R/z})$$

Scaling function  $f_M$  only depends on ratio  $\epsilon/T \Rightarrow$  **universality**

Full **curve** : LSI-prediction, with  $y = t/s$  fixed and  $a = a' = 1/z$ .

first test of 'Galilei-invariance' for  $z \neq 2$  in a non-linear model

# Practical tests of LSI, III : autocorrelation



(a)  $\epsilon = 0.5, T = 1$ , (b,c)  $\epsilon = 2, T = 1$  (d)  $t = [200, 300, 500, 1000, 2000]$

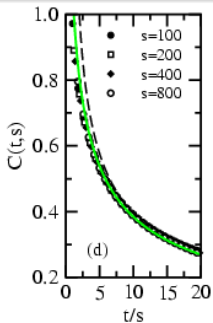
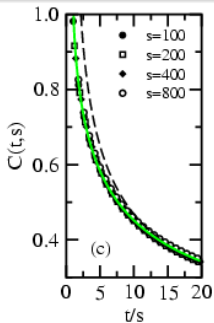
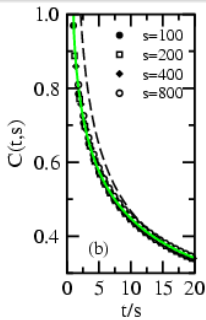
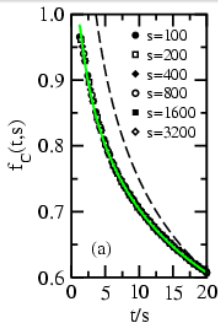
**No simple scaling with  $y = t/s$  for  $z \gtrsim 4$ !**

P, S & R 06; H & P 06

↘ indication for 'superageing'?

PAUL, SCHEHR, RIEGER 07

1. subtract finite-time correction,  $C(t,s) = f_C(t/s) - s^{-b'} g_C(t/s)$
  2. then scaling of  $C(t,s)$  according to **simple ageing** with  $y = t/s$
- \* Scaling function  $f_C$  only depends on ratio  $\epsilon/T \Rightarrow$  **universality**
  - \* ageing sets in at **late** time scale  $\tau = t - s \sim s^\zeta$  ZIPPOLD, KÜHN, HORNER 00
- use  $C(s + \tau, s; r) \sim \exp(-\nu r^2 s^{-2/z})$  generalised from OHTA, JASNOW, KAWASAKI 82



(a)  $\epsilon = 2, T = 1$  (b)  $\epsilon = T = 1$  (c)  $\epsilon = T = 0.5$  (d)  $\epsilon = 0.5, T = 1$

**Dashed line** : LSI with fully disordered initial correlator

Full **curve** : LSI prediction

$$f_C(y) = c_2 y^\rho \int_{\mathbb{R}^d} \frac{d\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^{2\beta} \exp\left(-\alpha |\mathbf{k}|^z (y-1) - \frac{\mathbf{k}^2}{4\nu}\right)$$

with  $\beta = \lambda_C - \lambda_R$ ,  $\rho = (2\beta + d - \lambda_C)/z$ . Used 'initial' correlator  $C(s + \tau, s; \mathbf{r}) \sim \exp(-\nu \mathbf{r}^2 s^{-2/z})$  : asymptotic, enough for  $z > 2$

Retour to the **non-disordered Ising model**, at  $T < T_c$  :  $z = 2$   
 requires a more precise form of the 'initial' correlator :

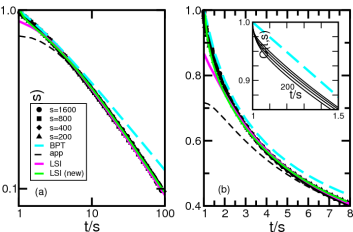
Ohta, Jasnow, Kawasaki '82

$$C(t, t; \mathbf{r}) = \frac{2}{\pi} \arcsin \left( \exp \left[ -\frac{\mathbf{r}^2}{L(t)^2} \right] \right)$$

hence autocorrelator in the scaling limit

$$C(y s, s) = C_0 y^\rho (y - 1)^{-\rho - \lambda_c / z} \int_0^\infty dx e^{-x} f_\nu \left( \sqrt{\frac{x}{y - 1}} \right)$$

$$f_\nu(\sqrt{u}) = \int_0^\infty dv \arcsin(e^{-\nu v}) J_0(\sqrt{u v})$$



of practical importance :  
 'good' choice of 'initial correlations'  
 $C_{\text{ini}}(\mathbf{r}) = c_0 \delta(\mathbf{r})$  not sufficient

## IV. Conclusions

- 1 look for extensions of dynamical scaling in ageing systems

recently, scaling derived for phase-ordering ARENZON ET AL. 07

- 2 here : **hypothesis** of **generalised Galilei-invariance**
- 3 leads to Bargman rule if  $z = 2$   
and further to 'integrability' if  $z \neq 1, 2$ .
- 4 **hidden** dynamical symmetry of **deterministic part** of (linear) Langevin equations
- 5 Tests : derive two-time response and correlation functions
- 6 LSI exactly proven for linear Langevin equations  
very good numerical evidence for non-linear systems

Some questions (the list could/should be extended) :

- how to physically justify Galilei-invariance ?
- how to extend to non-linear equations ? first attempts STOIMENOV & MH 05
- choice of the type of fractional derivative ?
- what is the algebraic (non-Lie!) structure of LSI ?
- treatment of master equations with LSI ?