Fractal dimension of domain walls in two-dimensional Ising spin glasses

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Outline

- Introduction
- Techniques
- Results
- Summary
**Model**

- \( N = L \times L \) Ising spins \( \sigma_i = \pm 1 \) on square lattice
- Periodic boundary conditions in one direction
- Edwards-Anderson Hamiltonian:

\[
\mathcal{H}(\sigma) = - \sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j
\]

interaction strength:

- \( J_{ij} > 0 \):
  - 
- \( J_{ij} < 0 \):
  - quenched disorder

frustration:
No energy gain via spin flip, i.e.

\[ \mathcal{H}(\sigma^*) \leq \mathcal{H}(\sigma) \]

Certain spin configuration for given disorder

Always: global spin flip connects GS pairs

Only:

\[ P(J_{ij}) \propto \exp\left(-J_{ij}^2/2\right) \quad P(J_{ij}) \propto [\delta(J_{ij}+1)+\delta(J_{ij}-1)] \]

trivial GS-degeneracy
umerous degenerate GS
Groundstate (GS)

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numerous degenerate GS
Domain Walls (DWs)

- Defined relative to 2 spin configurations (SCs) $\sigma^{(1)}/(2)$
- $\sigma^{(1)}$: 
- $\sigma^{(2)}$: 
- Separates regions of agreeing/disagreeing SCs

**DW energy:**

$$\Delta E = 2 \sum_{\langle ij \rangle \in \mathcal{D}} J_{ij} \sigma_i^{(1)} \sigma_j^{(1)}$$

$\mathcal{D} \equiv$ bonds satisfied by only 1 SC
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$D \equiv$ bonds satisfied by only 1 SC
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Construct weighted graph \( G = (V, E, \omega) \)

- \( V(G) \) elementary plaquettes (EP)
- \( E(G) \) connect EP with common side
- \( \omega \) energy contribution to DW

bonds satisfied in GS:

DW segment \( \omega \geq 0 \)
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bonds not satisfied in GS:

$\uparrow \quad \downarrow \quad \uparrow \quad \uparrow$

$\uparrow \quad \downarrow \quad \uparrow \quad \downarrow$

DW segment $\omega \leq 0$
Construct weighted graph $G = (V, E, \omega)$

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**Conservative weight function:**

$$\omega(C) = \sum_{\langle ij \rangle \in C} J_{ij} \sigma_i \sigma_j \geq 0$$
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**conservative weight function:**

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**DW:** shortest $(\text{top}, \text{bottom})$—path
Shortest Paths

- $G$: undirected graph allowing for negative edge weights
- Shortest path problem on dual requires matching techniques
  - i) Dual graph $\rightarrow$ auxiliary graph
  - ii) Find minimum weighted perfect matching (MWPM)
  - iii) Interpret MWPM as single pair shortest path (SPSP)
Obtain auxiliary graph $H$:
- Replace edge by path

Example dual graph $G$: 
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MW perfect matching

MWPM on $H$:
- Edmonds algorithm yields matching $M$
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Example
SPSP on $G$:
Remarks

- DWs $\rightarrow$ SPSP on dual graph with certain weight function
- Constraints yield DWs with extremal length
  - **Penalty:** $\omega \rightarrow \omega + \epsilon$ minimal length
  - **Reward:** $\omega \rightarrow \omega - \epsilon$ maximal length (only lower bound)
Typical DWs arising from gaussian disorder (left) and min./max. length DWs subject to bimodal disorder (middle/right)
Fractal dimension of domain walls

Scaling of fractal line:
\[
\langle \ell \rangle \propto L^{d_f} \quad 1 \leq d_f \leq 2
\]
\[
\langle r \rangle \propto L^{d_r} \quad d_r = 1
\]

Previous results on gaussian disorder:
\[d_f = 1.28(2) \text{ (Kawashima et al, cond-mat/9911120)}\]

<table>
<thead>
<tr>
<th></th>
<th>(d_f)</th>
<th>(d_r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss</td>
<td>1.271(1)</td>
<td>1.018(4)</td>
</tr>
<tr>
<td>± short</td>
<td>1.097(1)</td>
<td>1.012(3)</td>
</tr>
<tr>
<td>± long</td>
<td>1.396(10)</td>
<td>0.999(2)</td>
</tr>
</tbody>
</table>
Previous results on the DW entropy suggested different scaling of DWs with $E = 0$ (R. Fisch cond-mat/0608460)

<table>
<thead>
<tr>
<th>$E$</th>
<th>$d_f$ short DWs</th>
<th>$d_f$ long DWs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.095(1)</td>
<td>1.394(1)</td>
</tr>
<tr>
<td>2</td>
<td>1.102(1)</td>
<td>1.404(2)</td>
</tr>
</tbody>
</table>

No significant differences in $d_f$ for DWs with different energies
Scaled DW length distribution $P_L(\ell)$ for $40 \leq L \leq 160$. Left: gaussian, middle/right: bimodal bond distribution

\[
L^{d_f} P_L(\ell) = f\left(\frac{\ell - \langle \ell \rangle}{L^{d_f}}\right)
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Summary

- Groundstate study on 2D Ising spin glasses with short ranged interactions
- Shortest path approach to the problem of finding DWs
- Fractal dimension of DWs for different types of disorder distributions
Thank You!

Audience
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