# Fractal dimension of domain walls in two-dimensional Ising spin glasses


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# Outline

- Introduction
- Techniques
- Results
- Summary

# Model

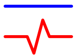
- $N = L \times L$ Ising spins $\sigma_i = \pm 1$ on square lattice
- Periodic boundary conditions in one direction
- Edwards-Anderson Hamiltonian:

$$\mathcal{H}(\sigma) = -\sum_{\langle ij \rangle} J_{ij}\sigma_i\sigma_j$$

interaction strength:

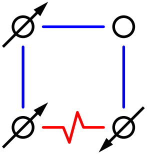
$J_{ij} > 0$ :

$J_{ij} < 0$ :



quenched disorder

frustration:

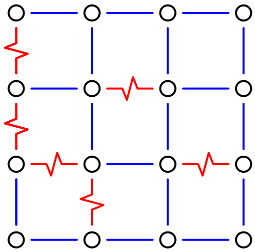


# Groundstate (GS)

- No energy gain via spin flip, i.e.

$$\mathcal{H}(\sigma^\star) \leq \mathcal{H}(\sigma)$$

- Certain spin configuration for given disorder



- Always: global spin flip connects GS pairs
- Only:

$P(J_{ij}) \propto \exp\left(-J_{ij}^2/2\right)$

trivial GS-degeneracy

$P(J_{ij}) \propto [\delta(J_{ij}+1)+\delta(J_{ij}-1)]$

numerous degenerate GS

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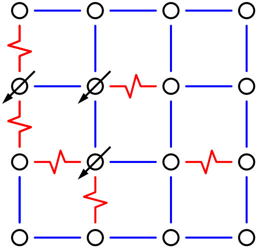

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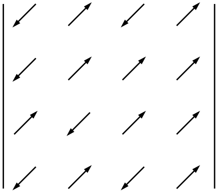
# Domain Walls (DWs)

- Defined relative to 2 spin configurations (SCs) $\sigma^{(1)/(2)}$
- $\sigma^{(1)}$:
  $\sigma^{(2)}$:
- Separates regions of agreeing/disagreeing SCs

  DW energy:

$$\Delta E = 2 \sum_{\langle ij \rangle \in \mathcal{D}} J_{ij} \sigma_i^{(1)} \sigma_j^{(1)}$$

$\mathcal{D} \equiv$ bonds satisfied by only 1 SC



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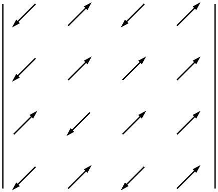

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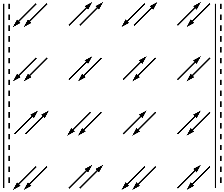

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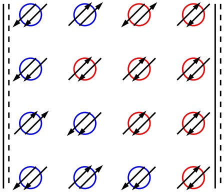

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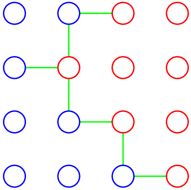

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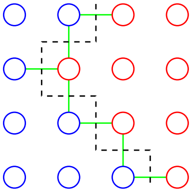
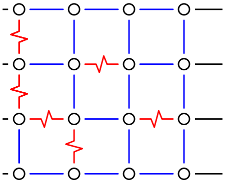


# Dual graph

- Construct weighted graph $G = (V, E, \omega)$

| | |
|---|---|
| $V(G)$ | elementary plaquettes (EP) |
| $E(G)$ | connect EP with common side |
| $\omega$ | energy contribution to DW |



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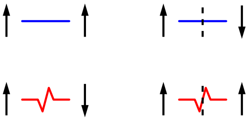

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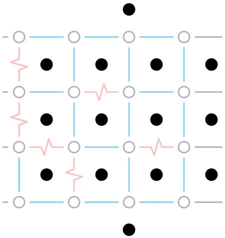

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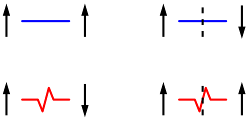

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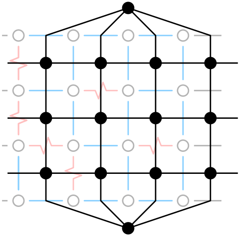

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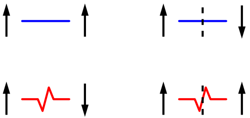

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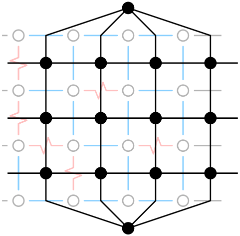
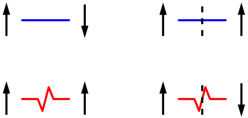


bonds not satisfied in GS:



DW segment $\omega \leq 0$

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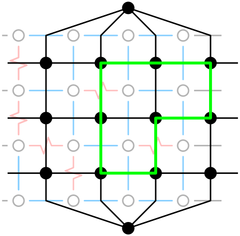

conservative weight function:

$$\omega(\mathcal{C}) = \sum_{\langle ij \rangle \in \mathcal{C}} J_{ij}\sigma_i\sigma_j \geq 0$$

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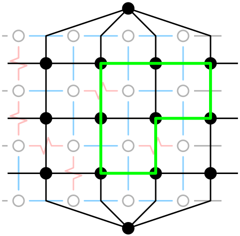

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- DW: shortest $(\mathrm{top}, \mathrm{bottom})-$path

# Shortest Paths

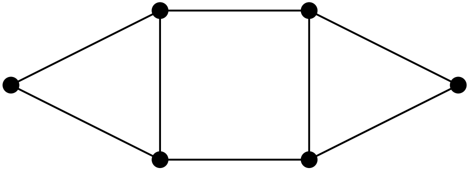
- $G$: undirected graph allowing for negative edge weights
- Shortest path problem on dual requires matching techniques
  i) Dual graph $\rightarrow$ auxiliary graph
  ii) Find minimum weighted perfect matching (MWPM)
  iii) Interpret MWPM as single pair shortest path (SPSP)

# Auxiliary graph

Obtain auxiliary graph $H$:

- Replace edge by path

- Example
  dual graph $G$:



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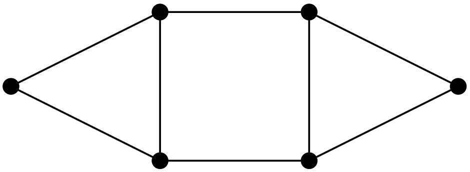

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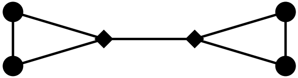

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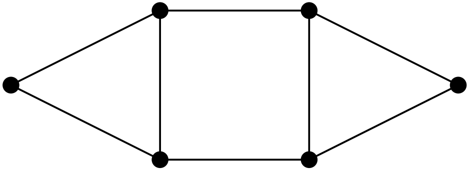

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Obtain auxiliary graph $H$:

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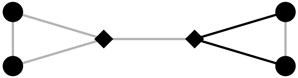
- Introduce weights



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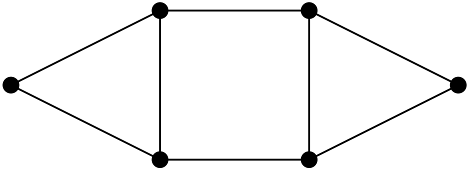

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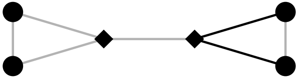

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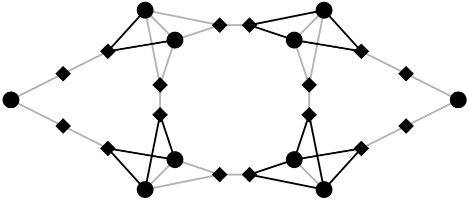

# MW perfect matching

MWPM on $H$:

- Edmonds algorithm yields matching $M$
- Polynomial running time
- Interpret $M$ as shortest path on $G$

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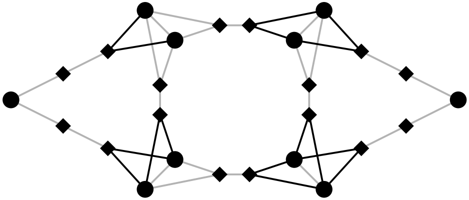

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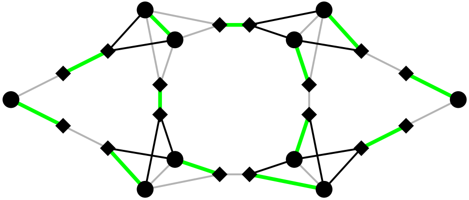

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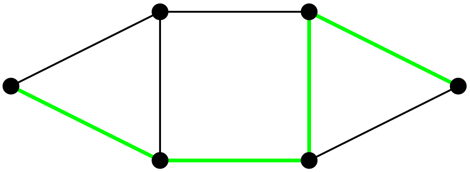
  SPSP on $G$:



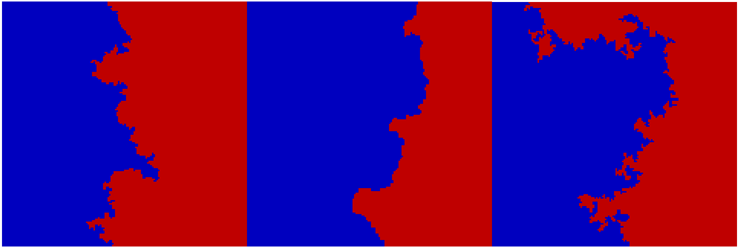
# Remarks

- DWs $\rightarrow$ SPSP on dual graph with certain weight function
- Constraints yield DWs with extremal lenght

  Penalty: $\omega \rightarrow \omega + \epsilon$ minimal length

  Reward: $\omega \rightarrow \omega - \epsilon$ maximal length (only lower bound)

# DW samples

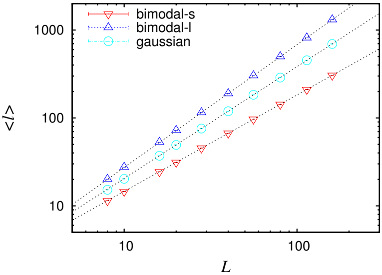


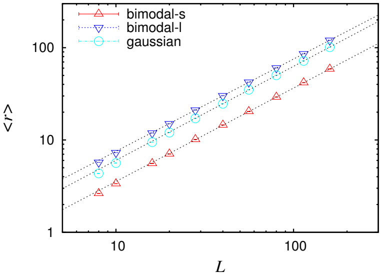
Typical DWs arising from gaussian disorder (left) and min./max. length DWs subject to bimodal disorder (middle/right)

# Fractal dimension of domain walls

- Scaling of fractal line: $\langle \ell \rangle \propto L^{d_f}$ $\quad 1 \le d_f \le 2$

  $\langle r \rangle \propto L^{d_r}$ $\quad d_r = 1$



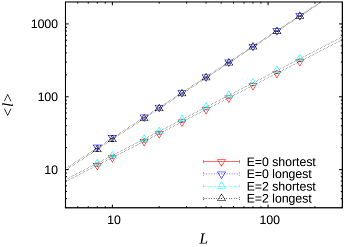

- Previous results on gaussian disorder:

  $d_f = 1.28(2)$ (Kawashima *et al* cond-mat/9911120)

| | $d_f$ | $d_r$ |
|---|---|---|
| Gauss | 1.271(1) | 1.018(4) |
| $\pm$ short | 1.097(1) | 1.012(3) |
| $\pm$ long | 1.396(10) | 0.999(2) |

# Bimodal case study

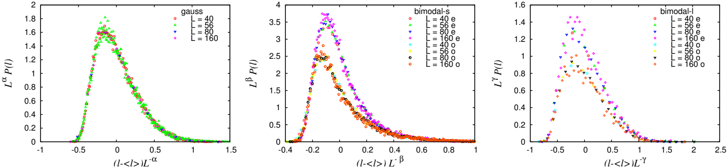
- Previous results on the DW entropy suggested different scaling of DWs with $E = 0$ (R. Fisch cond-mat/0608460)




| E | $d_f$ short DWs | $d_f$ long DWs |
|---|---|---|
| 0 | 1.095(1) | 1.394(1) |
| 2 | 1.102(1) | 1.404(2) |

- No significant differences in $d_f$ for DWs with different energies

# DW length




- Scaled DW length distribution $P_L(\ell)$ for $40 \leq L \leq 160$. Left: gaussian, middle/right: bimodal bond distribution

$$L^{d_f} P_L(\ell) = f\left(\frac{\ell - \langle \ell \rangle}{L^{d_f}}\right)$$

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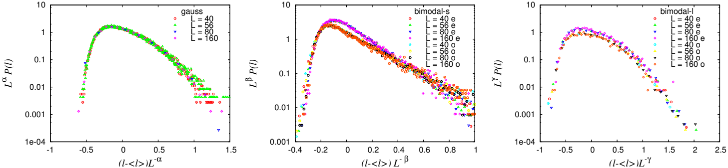


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# Summary

- Groundstate study on 2D Ising spin glasses with short ranged interactions
- Shortest path approach to the problem of finding DWs
- Fractal dimension of DWs for different types of disorder distributions

# Thank You!

- Audience
- A. K. Hartmann
- Financial support: VolkswagenStiftung