SCALING THEORY for LOGARITHMIC CORRECTION EXPONENTS

RK, D.A Johnston and W Janke, PRL 96, 115701 (2006);
97, 155702 (2006).

Background

We are interested in models of the type

$$Z = \sum_{\{s_i\}} e^{-\beta E + hM}$$

with $\beta = \frac{1}{k_B T}$, $h = \beta H$.

Thermodynamic functions are

Free energy $\quad f = \frac{1}{V} \ln Z$

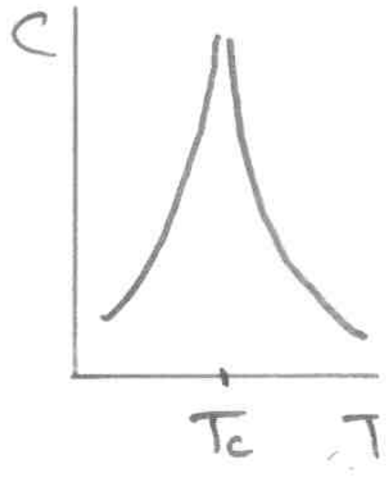
Specific heat $\quad C = \frac{1}{V} \langle (E - \langle E \rangle)^2 \rangle = \frac{\partial^2 f}{\partial \beta^2}$

Magnetization $\quad m = \frac{1}{V} \langle M \rangle = \frac{\partial f}{\partial h}$

Susceptibility $\quad \chi = \frac{1}{V} \langle (M - \langle M \rangle)^2 \rangle = \frac{\partial^2 f}{\partial h^2}$

At a second-order transition

$$C \sim |\beta - \beta_c|^{-\alpha} \quad \text{or} \quad |T - T_c|^{-\alpha}$$



i.e., $C \sim |t|^{-\alpha}$

CRITICAL EXPONENT

In fact,

Correlation length $\xi_\infty(t) \sim |t|^{-\nu}$

Specific heat $C_\infty(t) \sim |t|^{-\alpha}$

Susceptibility $\chi_\infty(t) \sim |t|^{-\gamma}$

Magnetization $\begin{cases} m_\infty(t) \sim |t|^{\beta} \\ m_\infty(h) \sim h^{\frac{1}{\delta}} \end{cases}$

SCALING RELATIONS

d = dimensionality, η = anomalous dimension

$\nu d = 2 - \alpha$ Widom (1965), Josephson (1967)

$2\beta + \gamma = 2 - \alpha$ Essam + Fisher (1963), Rushbrooke (1963)

$\beta(\delta - 1) = \gamma$ Widom (1964), Griffiths (1965)

$\nu(2 - \eta) = \gamma$ Fisher (1964)

THESE ARE FUNDAMENTALLY IMPORTANT IN STATISTICAL MECHANICS

LOGARITHMIC CORRECTIONS TO SCALING

In marginal scenarios there often exist logarithmic corrections:

Ex: $O(n)$ models interact in $d \leq 3$ and are trivial in $d \geq 5$.

$d = 4$ is marginal

Ex: Harris criterion $\Rightarrow$ if $\alpha > 0$ for pure system, exponents may change if system is disordered. If $\alpha < 0$, no change expected. $\alpha = 0$ is marginal

There, at $h = 0$,

$$\xi_\infty \sim |t|^{-\nu} |\ln t|^{\hat{\nu}}$$

$$C_\infty \sim |t|^{-\alpha} |\ln t|^{\hat{\alpha}}$$

$$\chi_\infty \sim |t|^{-\gamma} |\ln t|^{\hat{\gamma}}$$

$$m_\infty \sim |t|^{\beta} |\ln t|^{\hat{\beta}}$$

$$\langle s(0) s(x) \rangle \sim x^{-(d-2+\eta)} (\ln x)^{\hat{\eta}} D\left(\frac{x}{\xi_\infty}\right)$$

and at $t = 0$,

$$m_\infty \sim h^{\frac{1}{\delta}} |\ln h|^{\hat{\delta}}$$

It may also happen that $\xi_L(t=0) \sim L (\ln L)^{\hat{q}}$

IDEA: Since ∃ relationships between leading exponents, maybe there also exists relations between the exponents of the logs.

PLAN: Derive 4 relations between 7 exponents

Then: Confront them with literature

And: Make predictions.

Table 1: Hitherto known exponents for various models

Pure Ising	$\alpha = 0$	$\beta = \frac{1}{8}$	$\gamma = \frac{7}{4}$	$\delta = 15$	$\nu = 1$	$\Delta = \frac{15}{8}$		$\eta = 0$
in $d = 2$:	$\hat\alpha = 1$	$\hat\beta = 0$	$\hat\gamma = 0$	$\hat\delta = 0$	$\hat\nu = 0$	$\hat\Delta = 0$	$\hat q = 0$	$\hat\eta = 0$
Random	$\alpha = 0$	$\beta = \frac{1}{8}$	$\gamma = \frac{7}{4}$	$\delta = 15$	$\nu = 1$	$\Delta = \frac{15}{8}$		$\eta = 0$
Ising ($d = 2$):	$\hat\alpha = 0$	$\hat\beta = -\frac{1}{16}$	$\hat\gamma = \frac{7}{8}$	$\hat\delta = 0$	$\hat\nu = \frac{1}{2}$	$\hat\Delta =$?	$\hat q =$?	$\hat\eta = 0$
$O(n)$ ϕ^4_4:	$\alpha = 0$	$\beta = \frac{1}{2}$	$\gamma = 1$	$\delta = 3$	$\nu = \frac{1}{2}$	$\Delta = \frac{3}{2}$		$\eta = 0$
	$\hat\alpha = \frac{4-n}{n+8}$	$\hat\beta = \frac{3}{n+8}$	$\hat\gamma = \frac{n+2}{n+8}$	$\hat\delta = \frac{1}{3}$	$\hat\nu = \frac{n+2}{2(n+8)}$	$\hat\Delta = \frac{1-n}{n+8}$	$\hat q = \frac{1}{4}$	$\hat\eta = 0$
$q = 4$ Potts	$\alpha = \frac{2}{3}$	$\beta = \frac{1}{12}$	$\gamma = \frac{7}{6}$	$\delta = 15$	$\nu = \frac{2}{3}$	$\Delta = \frac{5}{4}$		$\eta = \frac{1}{4}$
in $d = 2$:	$\hat\alpha = -1$	$\hat\beta = -\frac{1}{8}$	$\hat\gamma = \frac{3}{4}$	$\hat\delta = -\frac{1}{15}$	$\hat\nu = \frac{1}{2}$	$\hat\Delta =$?	$\hat q = 0$	$\hat\eta = -\frac{1}{8}$
Long-range	$\alpha = 0$	$\beta = \frac{1}{2}$	$\gamma = 1$	$\delta = 3$	$\nu = \frac{1}{\sigma}$	$\Delta = \frac{3}{2}$		$\eta = 2 - \sigma$
($d_c = 2\sigma$):	$\hat\alpha = \frac{4-n}{n+8}$	$\hat\beta = \frac{3}{n+8}$	$\hat\gamma = \frac{n+2}{n+8}$	$\hat\delta = \frac{1}{3}$	$\hat\nu = \frac{n+2}{\sigma(n+8)}$	$\hat\Delta =$?	$\hat q =$?	$\hat\eta = 0$
m-comp	$\alpha = -1$	$\beta = 1$	$\gamma = 1$	$\delta = 2$	$\nu = \frac{1}{2}$	$\Delta = 1$		$\eta = 0$
spin glasses:	$\hat\alpha = -\frac{3m+1}{2m-1}$	$\hat\beta =$?	$\hat\gamma = \frac{2m}{2m-1}$	$\hat\delta =$?	$\hat\nu = \frac{5m}{6(2m-1)}$	$\hat\Delta =$?	$\hat q =$?	$\hat\eta =$?
Percolation:	$\alpha = -1$	$\beta = 1$	$\gamma = 1$	$\delta = 2$	$\nu = \frac{1}{2}$	$\Delta = 1$		$\eta = 0$
in $d = 6$:	$\hat\alpha = \frac{2}{7}$	$\hat\beta = \frac{2}{7}$	$\hat\gamma = \frac{2}{7}$	$\hat\delta = \frac{2}{7}$	$\hat\nu = \frac{5}{42}$	$\hat\Delta =$?	$\hat q =$?	$\hat\eta = \frac{1}{21}$
Lattice animals:	$\alpha = -1$	$\beta = 1$	$\gamma = 1$	$\delta = 2$	$\nu = \frac{1}{2}$	$\Delta = 1$		$\eta = 0$
in $d = 8$ (edge):	$\hat\alpha = -\frac{2}{3}$	$\hat\beta =$?	$\hat\gamma = \frac{2}{3}$	$\hat\delta =$?	$\hat\nu = \frac{5}{18}$	$\hat\Delta =$?	$\hat q =$?	$\hat\eta =$?

Also the N-colour Ashkin-Teller model, the $d = 3$ dipolar Ising and $d = 2$ XY models, ...

Tactics: Partition Function Zeros

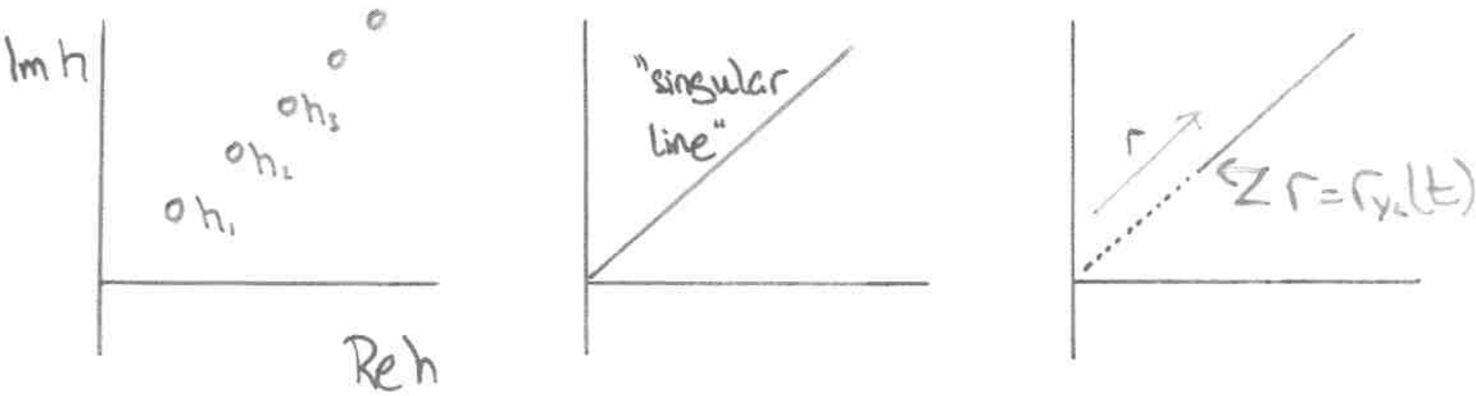
Phase transition = point of non-analyticity

Lee + Yang (1952) — Let $h \in \mathbb{C}$

$$\underset{\infty}{\overset{\uparrow}{f}} \;\Leftarrow\; = \frac{1}{V} \ln \underset{0}{\overset{\uparrow}{Z}}$$

Zeros in complex h $\longleftrightarrow$ Lee-Yang zeros

Zeros in complex t $\longleftrightarrow$ Fisher zeros (1964)



L = finite

$L = \infty$
$t = 0$

$L = \infty$
$t > 0$

Further assumption: $r_{YL}(t) \sim t^{\Delta} |\ln t|^{\hat{\Delta}}$

CALCULATION

$$Z \propto \prod_j (h - h_j)$$

$$\Rightarrow \quad f = L^{-d} \ln Z = L^{-d} \sum_j \ln(h - h_j)$$

$$\underset{L\to\infty}{\Longrightarrow} \quad f_\infty \sim \int_{r_{YL}(t)}^{\infty} g(r,t) \ln[h - h(r)]\, dr$$

Input χ_∞ $\longrightarrow$ form for density of zeros

Input m_∞ $\longrightarrow$ $\beta(\delta - 1) = \gamma$ [Griffiths']

$\hat{\beta}(\delta - 1) = \delta\hat{\delta} - \hat{\gamma}$ NEW

Input C_∞ $\longrightarrow$ $2\beta + \gamma = 2 - \alpha$ [Rushbrooke's]

$2\hat{\beta} - \hat{\gamma} = d\hat{q} - d\hat{\nu}$ NEW

Input FSS $\longrightarrow$ $\nu d = 2 - \alpha$ [Josephson's]

$\hat{\alpha} = d\hat{q} - d\hat{\nu}$ NEW

Other considerations $\longrightarrow$ $\gamma = \nu(2 - \eta)$ [Fisher's]

$\hat{\eta} = \hat{\gamma} - \hat{\nu}(2 - \eta)$ NEW

NEXT: Confront with literature

Ex: $O(n)\ \phi^4_4$

$\alpha = 0 \qquad \beta = \frac{1}{2} \qquad \gamma = 1 \qquad \delta = 3 \qquad \nu = \frac{1}{2}$

$\hat{\alpha} = \frac{4-n}{n+8} \qquad \hat{\beta} = \frac{3}{n+8} \qquad \hat{\gamma} = \frac{n+2}{n+8} \qquad \hat{\delta} = \frac{1}{3} \qquad \hat{\nu} = \frac{n+2}{2(n+8)} \qquad \hat{q} = \frac{1}{2}$

All old relations hold (of course)

All new relations hold too $\forall\, n$.

Ex: $q = 4$ Potts in $d = 2$

$\alpha = \frac{2}{3} \qquad \beta = \frac{1}{12} \qquad \gamma = \frac{7}{6} \qquad \delta = 15 \qquad \nu = \frac{2}{3} \qquad \eta = \frac{1}{4}$

$\hat{\alpha} = -1 \qquad \hat{\beta} = \frac{-1}{8} \qquad \hat{\gamma} = \frac{3}{4} \qquad \hat{\delta} = -\frac{1}{15} \qquad \hat{\nu} = \frac{1}{2} \qquad \hat{\eta} = -\frac{1}{8} \qquad \hat{q} = 0$

All relations hold.

Ex: Percolation in $d = 6$

m-component spin glasses in $d = 6$

Lattice animals in $d = 8$

Models with long-range interactions in low

All relations hold

and we make new predictions

Table 2: Now known exponents for various models

Pure	$\alpha = 0$	$\beta = \frac{1}{8}$	$\gamma = \frac{7}{4}$	$\delta = 15$	$\nu = 1$	$\Delta = \frac{15}{8}$		$\eta = 0$
Ising$_2$:	$\hat{\alpha} = 1$	$\hat{\beta} = 0$	$\hat{\gamma} = 0$	$\hat{\delta} = 0$	$\hat{\nu} = 0$	$\hat{\Delta} = 0$	$\hat{q} = 0$	$\hat{\eta} = 0$
Random	$\alpha = 0$	$\beta = \frac{1}{8}$	$\gamma = \frac{7}{4}$	$\delta = 15$	$\nu = 1$	$\Delta = \frac{15}{8}$		$\eta = 0$
Ising$_2$:	$\hat{\alpha} = 0$	$\hat{\beta} = -\frac{1}{16}$	$\hat{\gamma} = \frac{7}{8}$	$\hat{\delta} = 0$	$\hat{\nu} = \frac{1}{2}$	$\hat{\Delta} = -\frac{15}{16}$	$\hat{q} = 0$	$\hat{\eta} = 0$
$O(n)$ ϕ^4_4:	$\alpha = 0$	$\beta = \frac{1}{2}$	$\gamma = 1$	$\delta = 3$	$\nu = \frac{1}{2}$	$\Delta = \frac{3}{2}$		$\eta = 0$
	$\hat{\alpha} = \frac{4-n}{n+8}$	$\hat{\beta} = \frac{3}{n+8}$	$\hat{\gamma} = \frac{n+2}{n+8}$	$\hat{\delta} = \frac{1}{3}$	$\hat{\nu} = \frac{n+2}{2(n+8)}$	$\hat{\Delta} = \frac{1-n}{n+8}$	$\hat{q} = \frac{1}{4}$	$\hat{\eta} = 0$
Potts:	$\alpha = \frac{2}{3}$	$\beta = \frac{1}{12}$	$\gamma = \frac{7}{6}$	$\delta = 15$	$\nu = \frac{2}{3}$	$\Delta = \frac{5}{4}$		$\eta = \frac{1}{4}$
	$\hat{\alpha} = -1$	$\hat{\beta} = -\frac{1}{8}$	$\hat{\gamma} = \frac{3}{4}$	$\hat{\delta} = -\frac{1}{15}$	$\hat{\nu} = \frac{1}{2}$	$\hat{\Delta} = -\frac{7}{8}$	$\hat{q} = 0$	$\hat{\eta} = -\frac{1}{8}$
LR:	$\alpha = 0$	$\beta = \frac{1}{2}$	$\gamma = 1$	$\delta = 3$	$\nu = \frac{1}{\sigma}$	$\Delta = \frac{3}{2}$		$\eta = 2 - \sigma$
	$\hat{\alpha} = \frac{4-n}{n+8}$	$\hat{\beta} = \frac{3}{n+8}$	$\hat{\gamma} = \frac{n+2}{n+8}$	$\hat{\delta} = \frac{1}{3}$	$\hat{\nu} = \frac{n+2}{\sigma(n+8)}$	$\hat{\Delta} = \frac{1-N}{N+8}$	$\hat{q} = \frac{1}{2\sigma}$	$\hat{\eta} = 0$
m-	$\alpha = -1$	$\beta = 1$	$\gamma = 1$	$\delta = 2$	$\nu = \frac{1}{2}$	$\Delta = 1$		$\eta = 0$
SG:	$\hat{\alpha} = -\frac{3m+1}{2m-1}$	$\hat{\beta} = \frac{1+m}{2(1-2m)}$	$\hat{\gamma} = \frac{2m}{2m-1}$	$\hat{\delta} = \frac{1-3m}{4(1-2m)}$	$\hat{\nu} = \frac{5m}{6(2m-1)}$	$\hat{\Delta} = \frac{1+5m}{2(1-2m)}$	$\hat{q} = \frac{1}{6}$	$\hat{\eta} = \frac{m}{3(2m-}$
Perc:	$\alpha = -1$	$\beta = 1$	$\gamma = 1$	$\delta = 2$	$\nu = \frac{1}{2}$	$\Delta = 1$		$\eta = 0$
	$\hat{\alpha} = \frac{2}{7}$	$\hat{\beta} = \frac{2}{7}$	$\hat{\gamma} = \frac{2}{7}$	$\hat{\delta} = \frac{2}{7}$	$\hat{\nu} = \frac{5}{42}$	$\hat{\Delta} = 0$	$\hat{q} = \frac{1}{6}$	$\hat{\eta} = \frac{1}{21}$
YL:	$\alpha = -1$	$\beta = 1$	$\gamma = 1$	$\delta = 2$	$\nu = \frac{1}{2}$	$\Delta = 1$		$\eta = 0$
	$\hat{\alpha} = -\frac{2}{3}$	$\hat{\beta} = 0$	$\hat{\gamma} = \frac{2}{3}$	$\hat{\delta} = \frac{1}{3}$	$\hat{\nu} = \frac{5}{18}$	$\hat{\Delta} = -\frac{2}{3}$	$\hat{q} = \frac{1}{6}$	$\hat{\eta} = \frac{1}{9}$

Also the N-colour Ashkin-Teller model, etc ...

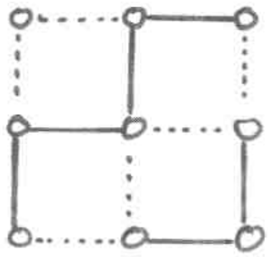
However...

the pure Ising model in $d=2$ dimensions has $C \sim \big|\ln|t|\big|$ and no other logarithms.

i.e., $\hat{\alpha} = 1$ and all other corrections vanish.

$\Rightarrow$ The new relation $\hat{\alpha} = d\hat{q} - d\hat{\nu}$ fails!!

The $d=2$ random-bond Ising model has



Shalaev, Shankar, Ludwig Jug (SSLJ)

$$\alpha = 0 \qquad \hat{\alpha} = 0 \longrightarrow C \sim \ln\ln|t|$$

(Dotsenko + Dotsenko)

$$\beta = 1/8 \qquad \hat{\beta} = -1/16$$

$$\gamma = 7/4 \qquad \hat{\gamma} = 7/8$$

$$\delta = 15 \qquad \hat{\delta} = 0$$

$$\nu = 1 \qquad \hat{\nu} = 1/2$$

$$\eta = 1/4 \qquad \hat{\eta} = 0$$

For SSLJ values, 3 new scaling relations hold.

But $\hat{\alpha} = d\hat{q} - d\hat{\nu}$ fails again!

$$\underset{0}{\hat{\alpha}} \qquad \underset{0}{d\hat{q}} \qquad \underset{2\cdot\frac{1}{2}=1}{d\hat{\nu}}$$

Controversy regarding $C \sim \ln \ln |t|$

Ziegler (1985), Heuer (1992), Kim + Patrascioiu (1994)
Kühn + Mazzeo (2000), Kim (2000)

$\Rightarrow$ finite C especially for random site

Many more support $C \sim \ln \ln |t|$
including Talapov + Shchur (1995)
Roder, Adler, Janke (1999)
see also Berche, Chatelain (2004)
Berche, Shchur (2004)

So $\hat{\alpha} = d\hat{\eta} - d\hat{\nu}$ fails in pure and random Ising cases in $d=2$ where $\alpha = 0$.

But it holds in the Ising (and $O(n)$) case in $d=4$, where also $\alpha = 0$.

Nonetheless, source of problem is $\alpha = 0$.

Consider Fisher zeros:

$$Z_L(t) \propto \prod_{j}^{L^d} \left(t - t_j(L)\right)$$

$$\Rightarrow \quad f_L(t) = L^{-d} \sum_{j=1}^{L^d} \ln\left(t - t_j(L)\right)$$

$$\Rightarrow \quad C_L(t) = -L^{-d} \sum_{j} \frac{1}{\left(t - t_j(L)\right)^2}$$

$$\Rightarrow \quad C_L(0) = -L^{-d} \sum_{j=1}^{L^d} t_j^{-2}$$

FSS arguments $\leadsto$

$$t_j(L) = \left(\frac{j}{L^d}\right)^{\frac{1}{\nu}} \left[\ln \frac{j}{L^d}\right]^{\frac{\hat{\nu} - \hat{q}}{\nu}} e^{i\phi}$$

This gives for the specific heat

$$C_L(0) \sim L^{\frac{2}{\nu} - d} \sum_{j=1}^{L^d} \left(\frac{1}{j}\right)^{1 + \frac{\alpha}{\nu d}} \left[\ln j\right]^{2\frac{\hat{q} - \hat{\nu}}{\nu}} \cos 2\phi$$

For $\alpha \neq 0$ this recovers what we had before

But if $\alpha = 0$, consider this piece ...

Special case: $\alpha = 0$

Have special contribution $\sum_{j=1}^{L^d} \frac{1}{j}$

This is like $\int^{L^d} \frac{1}{x}\,dx = \ln L$

and gives an extra log!

Scaling relation for $\alpha = 0$ is modified to

$$\hat{\alpha} = 1 + d\hat{q} - d\hat{\nu}$$

This holds!

The $d=2$ n-colour Ashkin Teller model also has $\alpha = 0$. There

$\hat{\alpha} = \frac{-n}{n-2}$ $\hat{\beta} = \frac{-(n-1)}{8(n-2)}$ $\hat{\gamma} = \frac{7(n-1)}{4(n-2)}$ $\hat{\delta} = 0$ $\hat{\nu} = \frac{n-1}{n-2}$

$\hat{q} = 0$

The modified scaling relation holds there too!

What about the $d=4$ case?

Again

$$C \sim L^{\frac{2}{\nu}-d} \sum_j \left(\frac{1}{j}\right)^{1+\frac{\alpha}{\nu d}} [\ln j]^{2\frac{\hat{q}\cdot\hat{\nu}}{\nu}\cos 2\phi}$$

The extra log does not arise if $\cos 2\phi = 0$

i.e., if $2\phi = \frac{\pi}{2}$

if $\phi = \frac{\pi}{4}$

This is what happens in $O(n)\ \phi^4_4$

[Glasser, Privman, Schulman, 1986
RK, Lang 1993]

So no extra log there!

What about Dotsenko + Dotsenko's $C \sim \ln \ln |t|$?

Again,

$$C \sim L^{\frac{2}{\nu}-d} \sum_{j=1}^{L^d} \left(\frac{1}{j}\right)^{1+\frac{\alpha}{\nu d}} [\ln j]^{2\frac{\hat{q}-\hat{\nu}}{\nu}} \cos 2\phi$$

If $\alpha = 0$,

$$C \sim \sum_{j=1}^{L^d} \frac{1}{j} (\ln j)^{2\frac{\hat{q}-\hat{\nu}}{\nu}}$$

If $2(\hat{\nu}-\hat{q}) = \nu$ this is -1

Then $$C \sim \sum_{j=1}^{L^d} \frac{1}{j} \frac{1}{\ln j}$$

$$\sim \int^{L^d} \frac{1}{x} \frac{1}{\ln x} dx$$

$$\sim \int^{L^d} \frac{d \ln x}{\ln x} \sim \ln \ln L$$

This is what happens in SSLJ | DD.

SUMMARY

$$\hat{\beta}(\delta - 1) = \delta\hat{\delta} - \hat{\gamma}$$

$$2\hat{\beta} - \hat{\gamma} = d\hat{q} - d\hat{v}$$

$$\hat{\eta} = \hat{\gamma} - \hat{v}(2 - \eta)$$

$$\hat{\alpha} = \begin{cases} d\hat{q} - d\hat{v} \\ 1 + d\hat{q} - d\hat{v} & \text{if } \alpha = 0,\ \phi \neq \frac{\pi}{4} \end{cases}$$

and $C \sim h \sim h \sim |t|$ in very special cases.