Random-bond Potts model for large q

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Introduction

Effect of quenched disorder on phase transitions

- Continuous phase transitions

  For bond disorder: Harris criterion

  Pure system's fixed point is

  - stable for $\alpha < 0$
  - unstable for $\alpha > 0$

- Discontinuous phase transitions

  No general theory

  General trend: the size of the discontinuity (c.f. latent heat) is reduced by disorder.

  - In 2d rigorous result:
    for continuous bond disorder there is no phase coexistence (Aizenman and Wehr).
  - In 3d numerical results:
    weak disorder $\rightarrow$ discontinuous transition
    strong disorder $\rightarrow$ continuous transition

Previous results in 3d

- Experiment: isotropic-to nematic transition of nCB liquid crystal (Iannacchione et al 1993)
- MC simulation for the $q = 3$-state site-diluted Potts model (Uzelac et al 1995,Ballesteros et al 2000)
- MC simulation for the $q = 4$-state bond-diluted Potts model (Chatelain et al 2001, Janke et al 2005)

Critical (and tricritical) exponents of random Potts models

	perc.	$q = 2$	$q = 3$	$q = 4$	$q \to \infty$	$q \to \infty$ (tricr.)
ν	0.877	0.684(5)	0.690(5)	0.747(10)	?	?
β/ν	0.477	0.518(2)	0.539(2)	0.732(20)	?	?

Statistical mechanics of the Potts model for large q

- Hamiltonian:

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij}\delta(\sigma_i, \sigma_j), \quad \sigma_i = 0, 1, \cdots, q-1 \tag{1}$$

  $J_{ij} > 0$ are i.i.d. random variables.

  – Distribution of couplings:

$$P(J_{ij}) = p\delta(J(1+\delta) - J_{ij}) + (1-p)\delta(J(1-\delta) - J_{ij}), \quad p = 1/2 \tag{2}$$

  – strength of disorder $0 < \delta \leq 1$
  pure system: $\delta = 0$, diluted system $\delta = 1$.

- Random cluster representation (Kasteleyn and Fortuin, 1969)

$$Z = \sum_G q^{c(G)} \prod_{ij \in G} \left[q^{\beta J_{ij}} - 1 \right] \tag{3}$$

  G: subset of bonds, $c(G)$: number of connected components of G
  rescaled temperature: $T \to T \ln q = O(1)$ and $\beta \to \beta / \ln q = O(1)$

- for $q \to \infty$: $q^{\beta J_{ij}} \gg 1, \quad Z = \sum_{G \subseteq E} q^{\phi(G)}, \quad \phi(G) = c(G) + \beta \sum_{ij \in G} J_{ij}$

- This is dominated by the largest term, $\phi^* = \max_G \phi(G)$.

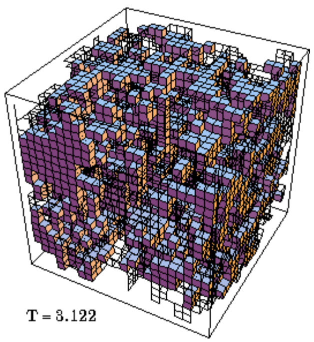
- the free energy is: $-\beta f = \phi^*/N$, N the number of sites of the lattice.

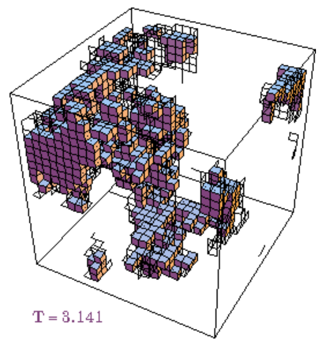
Optimal cooperation problem

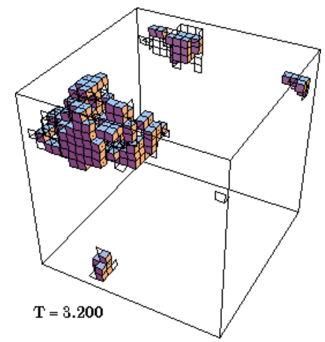
$$\boxed{\phi(G) = c(G) + \beta \sum_{ij \in G} J_{ij} = max}$$

- Approximate methods (simulated annealing, etc.) in 2d (24×24)(Juhász et al 2001)
- Combinatorial optimization algorithm for the submodular cost function (Anglès d'Auriac et al 2002)
- Application in 2d (512×512) (Anglès d'Auriac and Iglói 2003, Mercaldo et al 2004)
- Application in 3d ($40 \times 40 \times 40$) (Mercaldo et al 2005,2006)
- Application for complex networks (Karsai et al, 2007)

Connected parts of optimal sets in $3d$ at $\delta = .875$

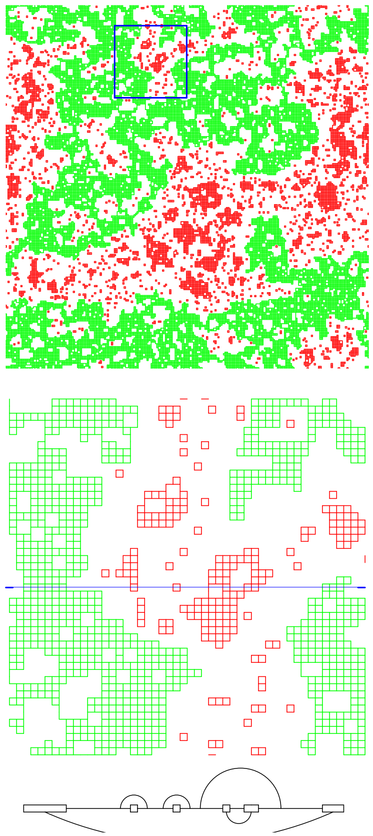






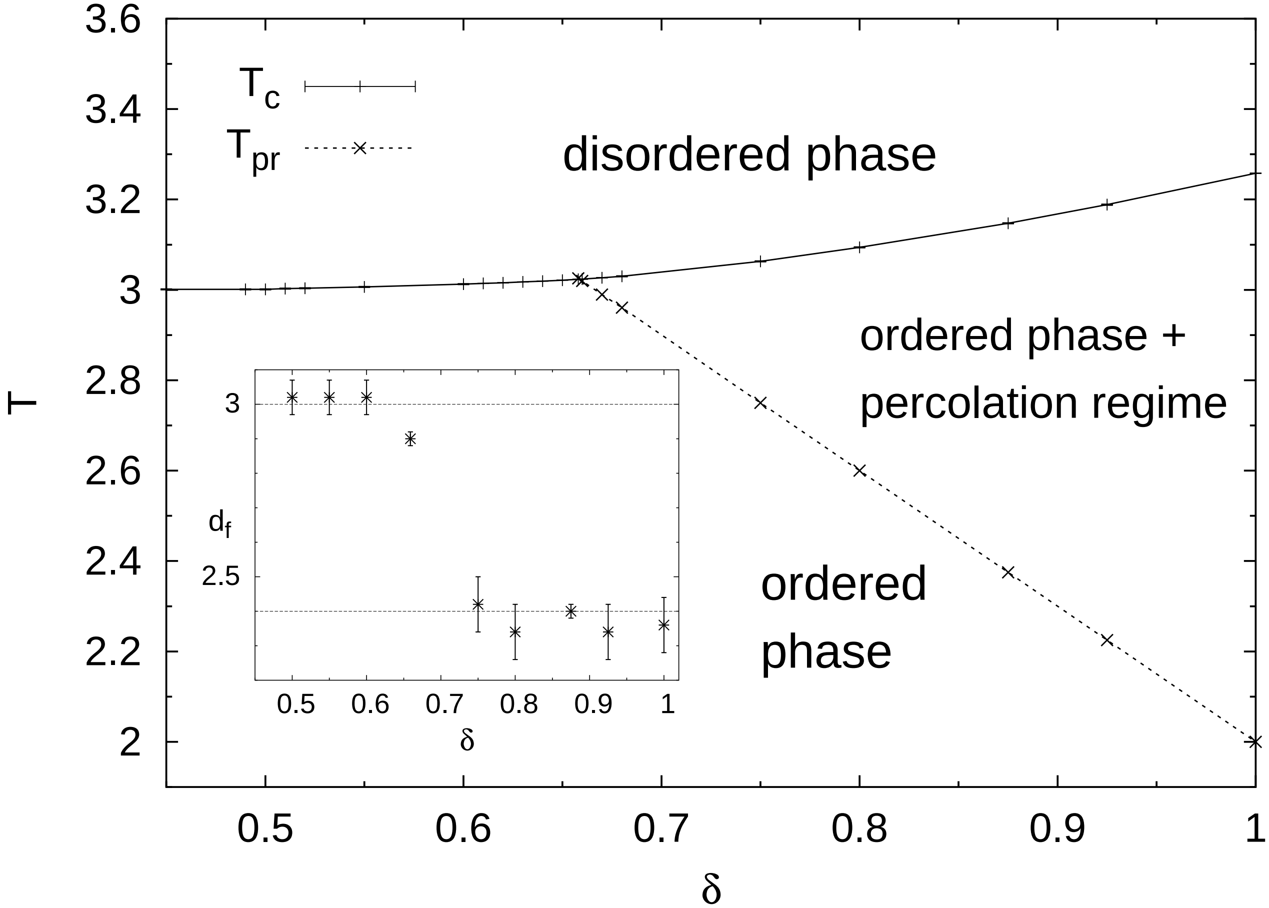
$T < T_c$ $T \approx T_c$ $T > T_c$

Illustration of the optimal set in $2d$ at the critical point



- the critical point is at $T_c = 2$ independently of δ.
- the largest connected cluster is a fractal.
- the fractal dimension d_f does not depend on δ.
- the critical exponents are conjectured exact (Anglès d'Auriac and Iglói 2003, Mercaldo et al 2004):
  - magnetization $x = d - d_f = (3 - \sqrt{5})/4 = .191$
  - surface magnetization $x_s = 1/2$
  - correlation length $\nu = 1$

Phase diagram in $3d$



Weak disorder

- Nonrandom system ($\delta = 0$) two homogeneous optimal sets:
  - $T < T_c(0)$ fully connected diagram, free-energy: $-\beta N f = 1 + N\beta J d$
  - $T > T_c(0)$ empty diagram, free-energy: $-\beta N f = N$

  $T_c(0) = Jd/(1 - 1/N)$, latent heat $\Delta e/T_c(0) = 1 - 1/N$

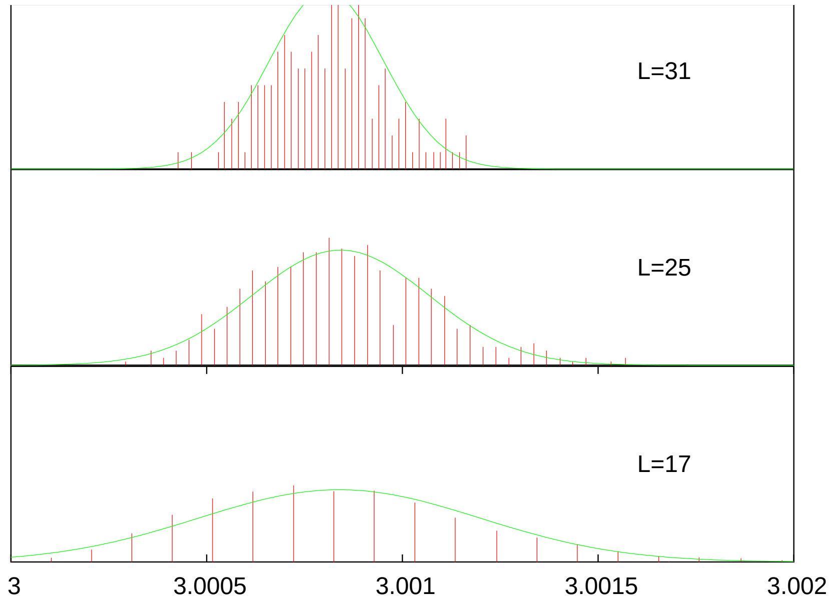
- Imry-Ma argument ($\delta \ll 1$)
  Creation in the disordered phase a connected cluster of size l. In the cost function:
  - gain, due to disorder fluctuations $\sim \delta l^{d/2}$
  - loss, due to creation of an interface $\sim l^{d-1}$
    - $d \leq 2$: infinite cluster is stable, no coexisting pure phases, $2nd$-order transition
    - $d > 2$: only finite clusters, coexisting pure phases, $1st$-order transition

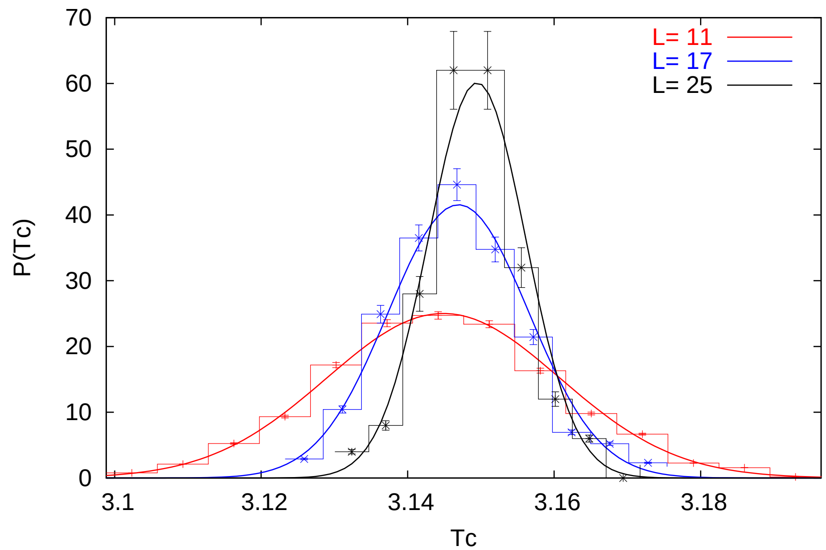
- Perturbational results

$$\ln(T_c - T_c(0)) \sim -\frac{1}{\delta^3}, \quad \ln(\Delta e/T_c - 1) \sim \ln(\Delta m - 1) \sim -\frac{1}{\delta^3} \tag{4}$$

Numerical results

Finite-size (percolation) temperatures





$\delta = .5$, $1st$-order

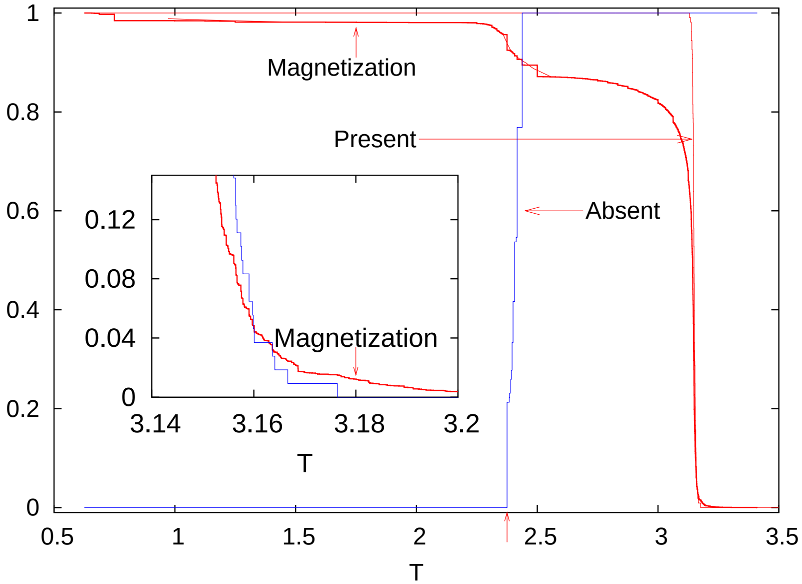
$\delta = .875$, $2nd$-order

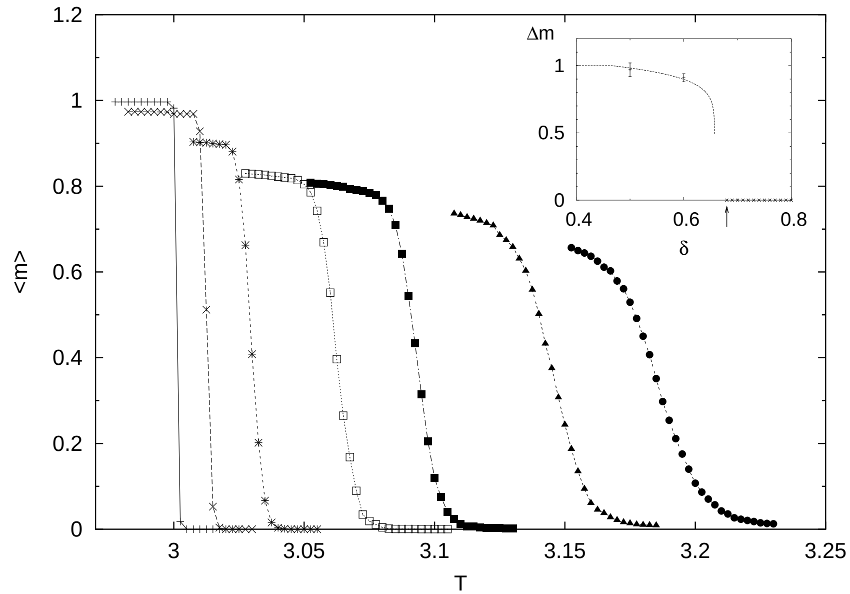
$$T_c^{av}(L) - T_c(\infty) \sim L^{-1/\tilde{\nu}}, \quad \Delta T_c(L) \sim L^{-1/\nu} \tag{5}$$

$\tilde{\nu} = 1/d, \quad \nu = 2/d$

$\tilde{\nu} = \nu.$

Magnetization and percolation probability





$\delta = .875$ $\delta = .5,\ .6,\ .68,\ .75,\ .8,\ .875,\ .925$

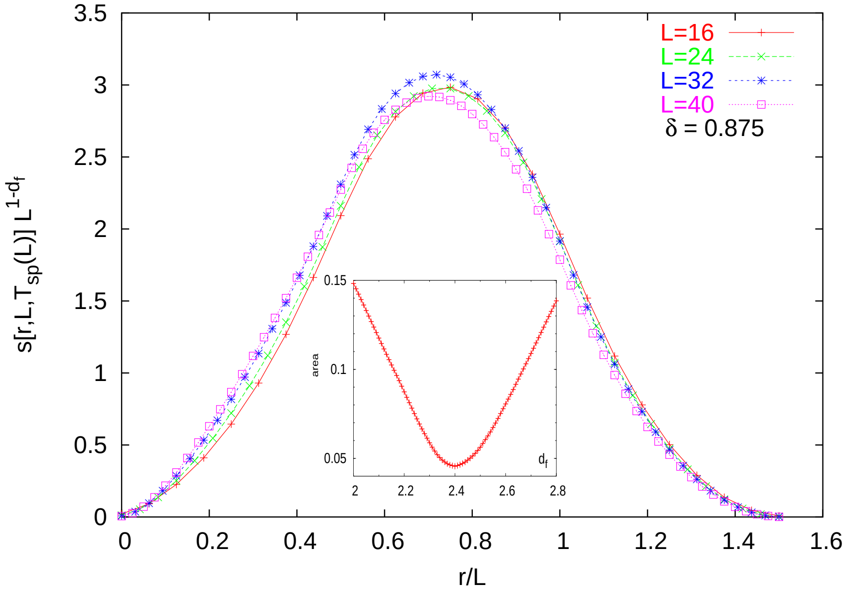
"Present": percolation probability of occupied bonds
"Absent": percolation probability of isolated sites

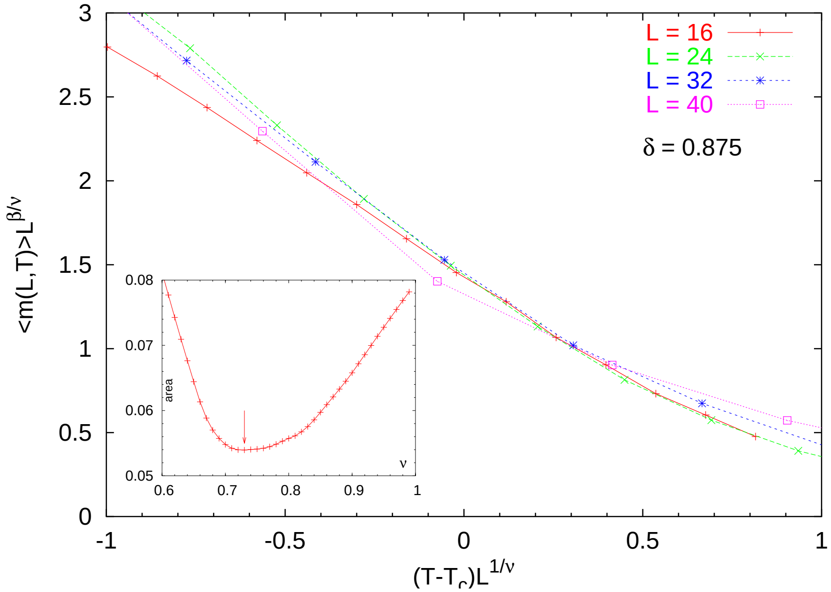
Region of percolation of isolated sites: $T_c > T > T_{pr}(\delta) = J_1 + 4J_2 = J(5 - 3\delta)$

Condition for the tricritical disorder: $\delta_t \geq \delta_{pr}$

Second-order transition regime

We measure the fractal dimension of the giant cluster, through $s(r, L, t)$: mass (number of points) in a shell around the reference point with unit width and radius, r.

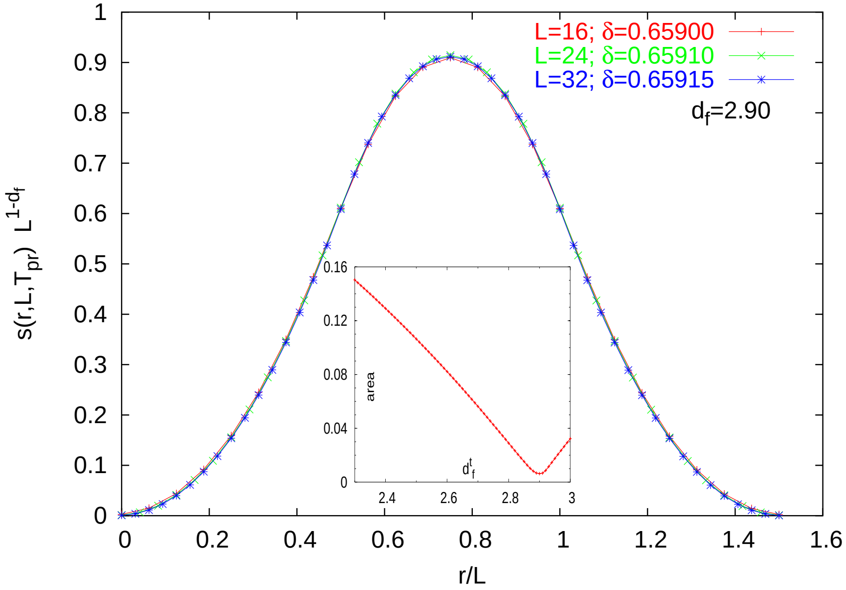


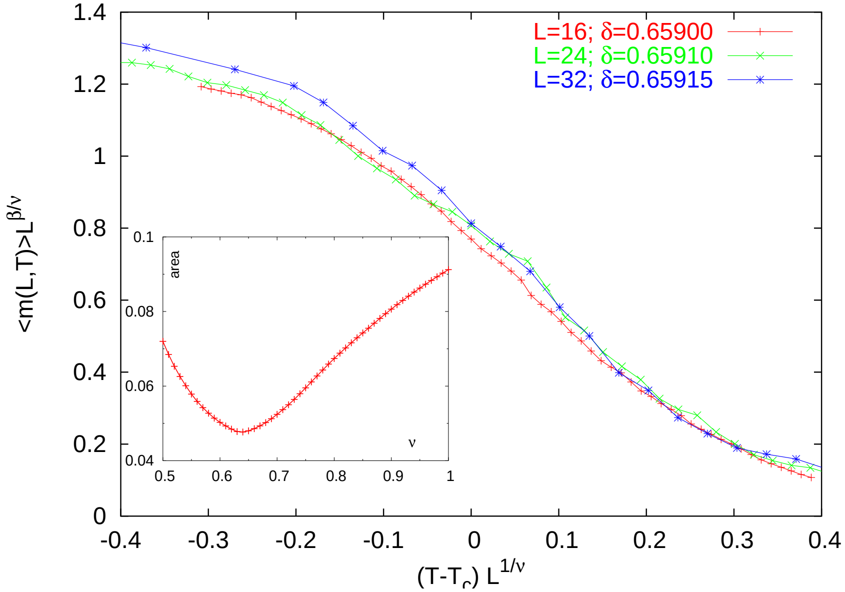


$s(r, L, t) = L^{d_f - 1}\tilde{s}(r/L, tL^{1/\nu})$
$d_f = 2.40(2)$

$m(t, L) = L^{x_m}\tilde{m}(tL^{1/\nu})$
$\nu = 0.73(2) > 2/d$

Tricritical transition





$d_f^t = 2.90(2), \quad x^t = 0.10(2)$

$\nu^t = 0.67(4) > 2/d$

Relation with the critical random-field Ising model

- In the RFIM there are two ordered phases (with up and dawn magnetizations), which are separated by an interface.
- In the RBPM there are two trivial optimal sets (empty and fully connected), which are also separated by an interface.
- In the solid-on-solid approximation the two interface Hamiltonians are equivalent in $d = 2 + \epsilon$ dimension. (Cardy and Jacobsen 1997)
- This is expected to hold in $d = 3$, too.
- Magnetization excitations in the critical RFIM are equivalent to energy-like excitations in the tricritical RBPM.
- exponent relation: $\nu^t = \nu^{RF}/(\beta^{RF} + \gamma^{RF}) \approx 0.67(2)$
- $\ln q$ is a dangerous irrelevant scaling variable thus x^t is hyperuniversal.

Conclusion

- disorder dependent 1st and 2nd order transition regimes
- 2nd order transition is related to percolation of empty sites
- accurate critical and tricritical exponents
- the tricritical exponents are probably hyperuniversal

Critical (and tricritical) exponents of random Potts models in $3d$

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β/ν	0.477	0.518(2)	0.539(2)	0.732(20)	0.60(2)	0.10(2)

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