Network dynamics: Laplacians of graphs and their local and global topology

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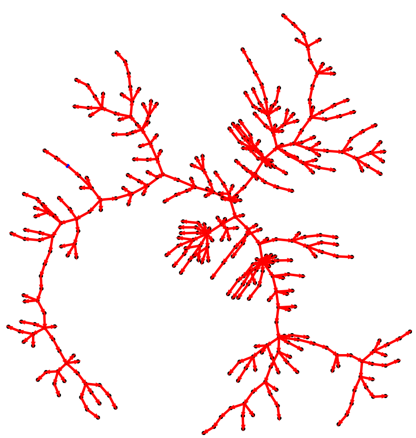
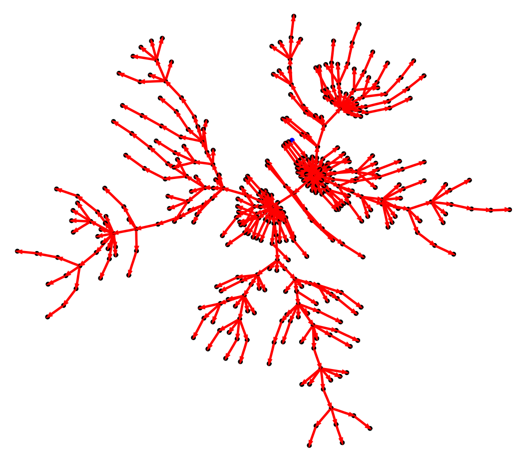
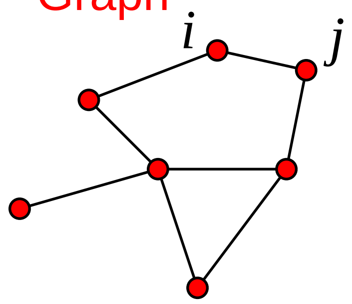




Dynamics of/on Networks

Graph



Neighbour relation $\langle ij \rangle$

Diffusion

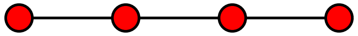
$$\frac{\partial \rho_j}{\partial t} = K \sum_{\langle ij \rangle} (\rho_i - \rho_j)$$

Harmonic excitation

$$\frac{\partial^2 x_j}{\partial t^2} = K \sum_{\langle ij \rangle} (x_i - x_j)$$

– can one hear the structure of the network?

Spectrum of the Laplacian



$$\sum_{\langle ij\rangle}(x_i - x_j) = \sum_j A_{ij}x_j \quad \text{with Laplacian } A_{ij} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

The Laplacian is

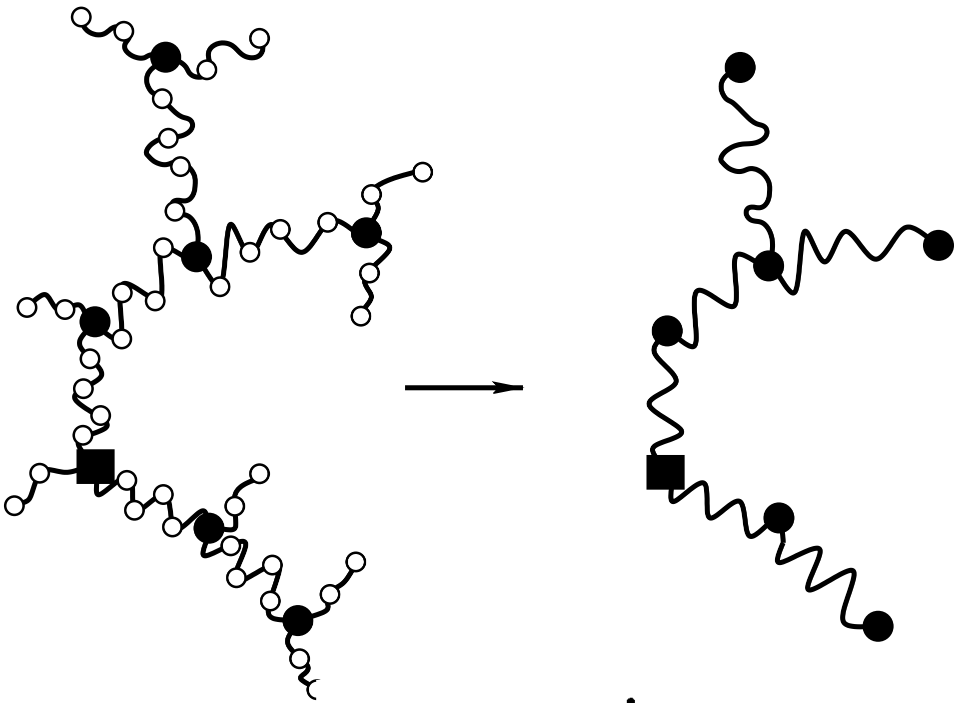
$$A_{ij} = \begin{cases} -1 & \text{if } \langle ij\rangle, \\ \sum_{\langle ik\rangle} 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

$\rho(\varepsilon)$: spectrum of eigenvalues of A_{ij}.
Random walk return probability :

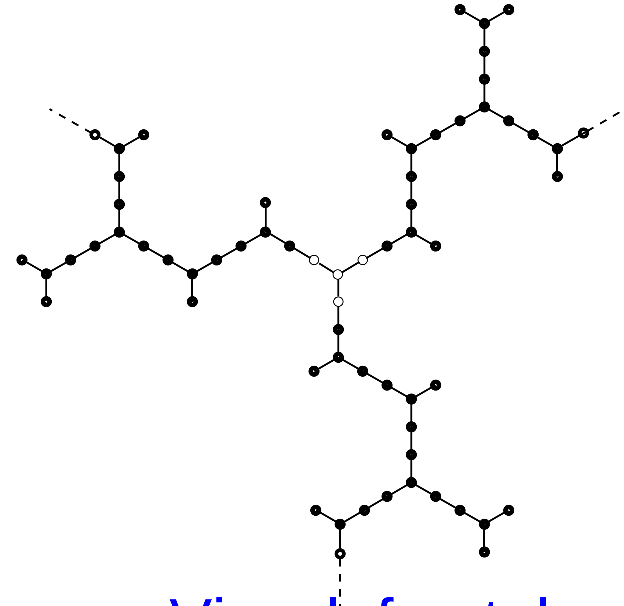
$$Q(t) = \int_{0+}^{\infty} d\varepsilon e^{-\varepsilon t} \rho(\varepsilon).$$

other quantities: replace $e^{-\varepsilon t}$ by some $f(\varepsilon, t)$.

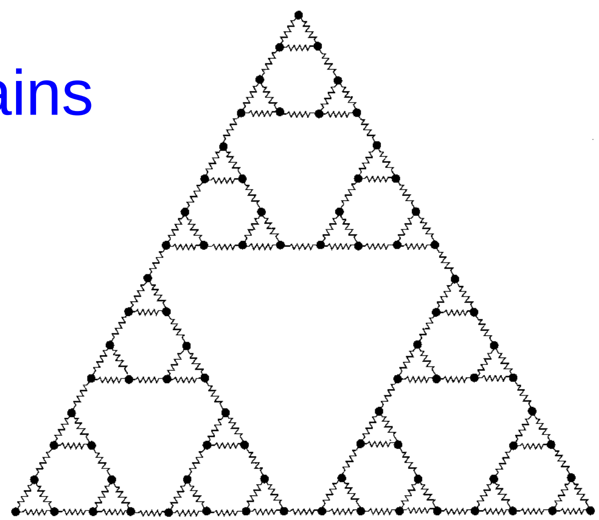
Repeated local structures



Spacer chains



Vicsek fractal

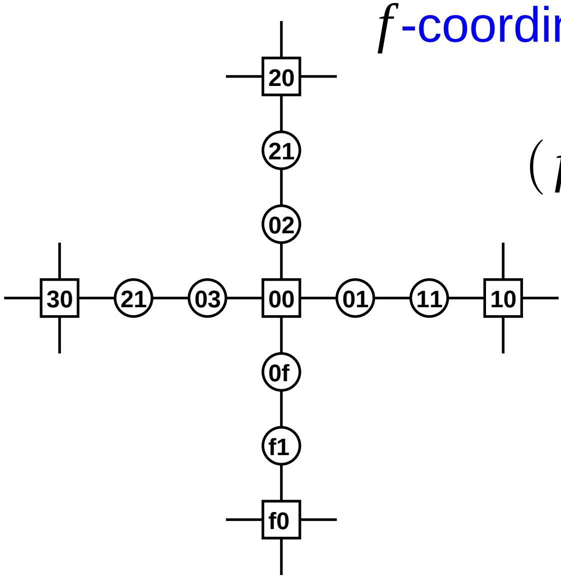


Dual Sierpinski fractal

Reduction (real space renormalization)

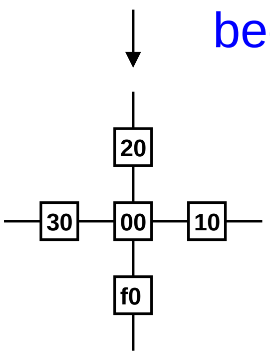
Eigenvalue equations

f-coordinated core:

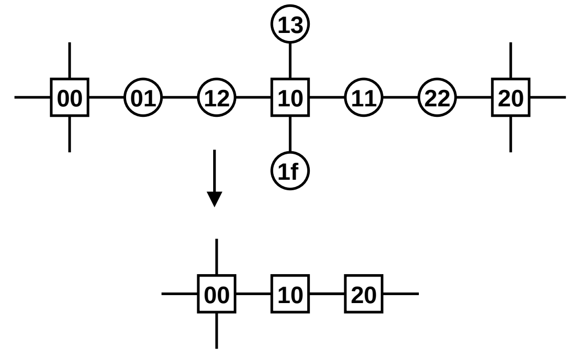


$$(f-\lambda)\Phi_j = \sum_{\alpha=1}^{f} \Phi_{i_\alpha}.$$

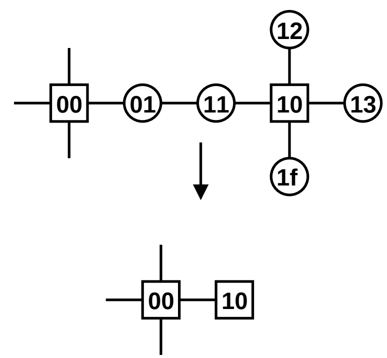
becomes:



$$(f-P(\lambda))\Phi_j = \sum_{\alpha=1}^{f} \Phi_{i_\alpha}.$$



$$(2-P(\lambda))\Phi_{i+1} = \Phi_i + \Phi_{i+2},$$

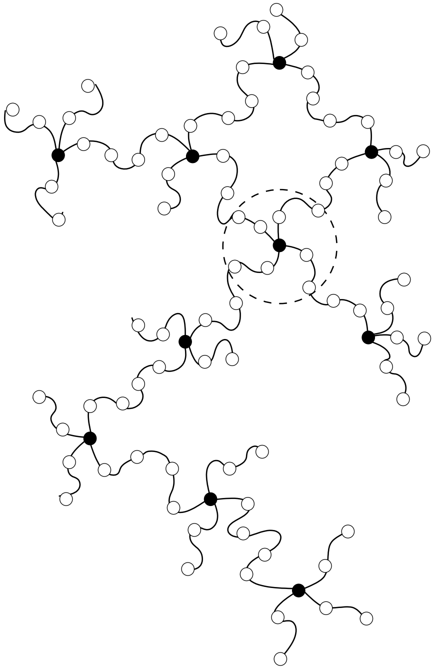


$$(1-P(\lambda))\Phi_k = \Phi_{k-1}.$$

$P(\lambda) = f + fU_{2k-1} - (f-\lambda)U_{2k}.$ with Chebyshev polynomials $U_n(1-\lambda/2)$.

Spectrum of decorated graph

Decoration by spacer and dangling chains



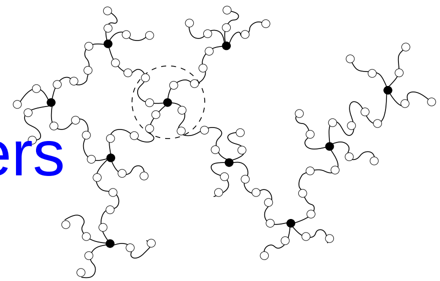
- eigenvalues derived from $\lambda_{\text{reduced}} \neq 0$:
  $\lambda_{\text{reduced}} = P(\lambda) = (U_k + U_{k-1})Q(\lambda)$
- eigenvalues derived from $\lambda_{\text{reduced}} = 0$:
  - global modes:
    $0 = Q(\lambda) = \lambda[(f-1)U_{k-1} + U_k]$
  - local modes:
    $0 = U_{k-1} + U_k$
    $0 = U_{k-1} - U_k$

  Chebyshev polynomials U_k. Spacer chain length $2k$.

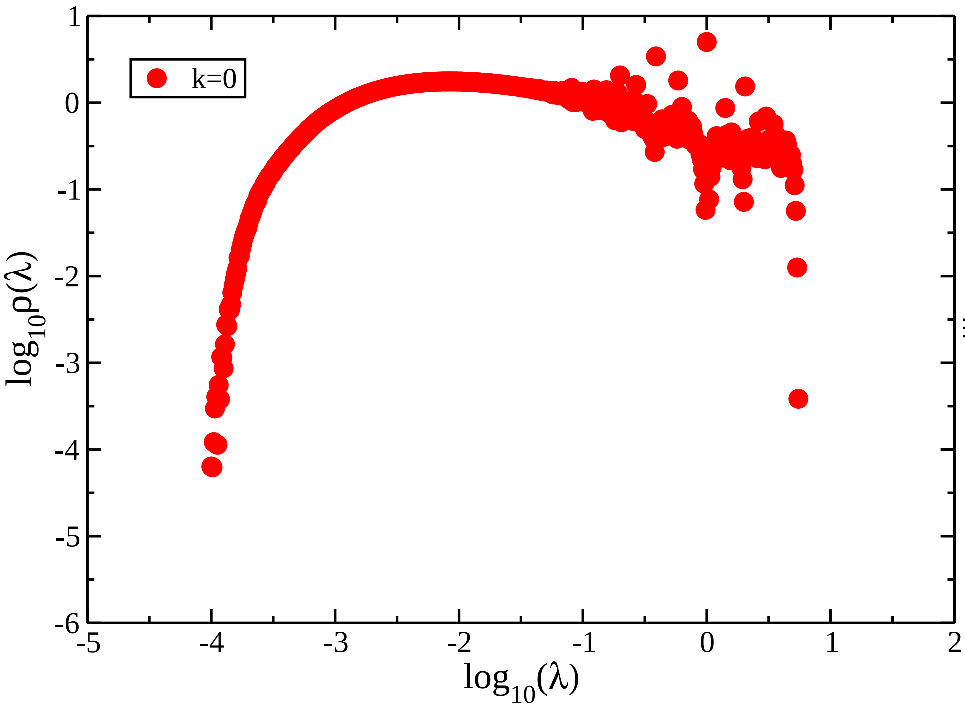
Spectrum of decorated graphs

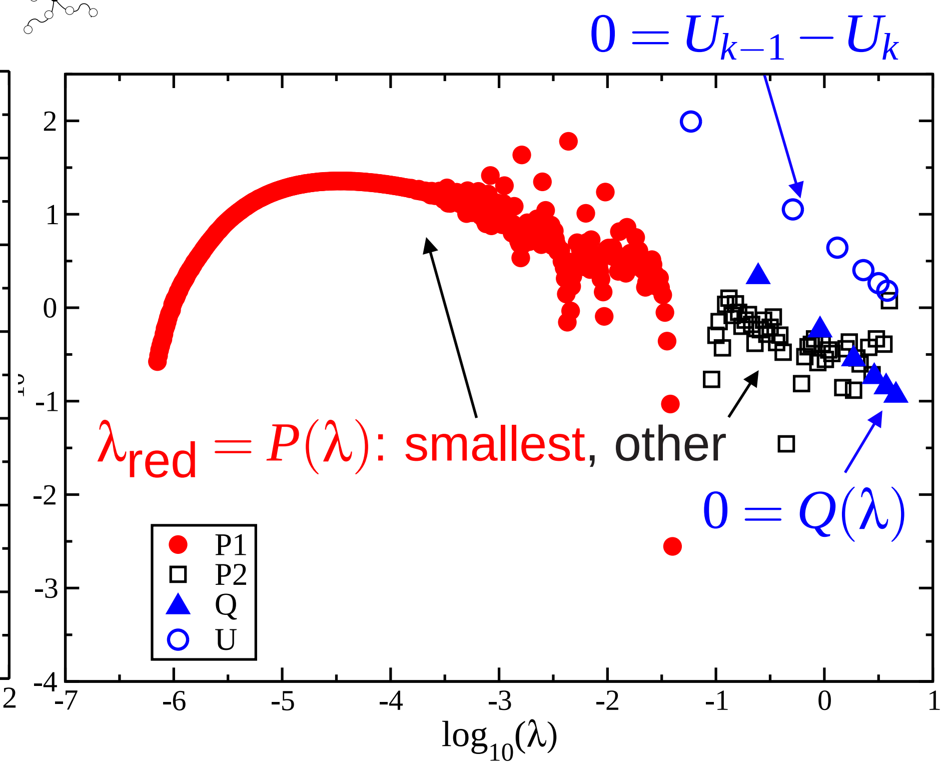
Decoration by spacer and dangling chains

bare branched polymers



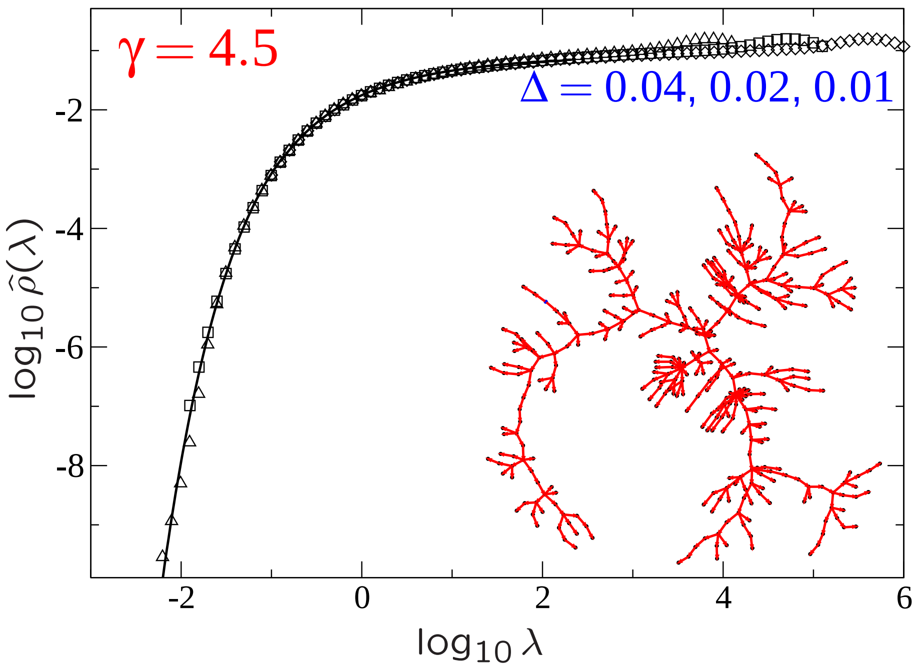
decorated b.p.

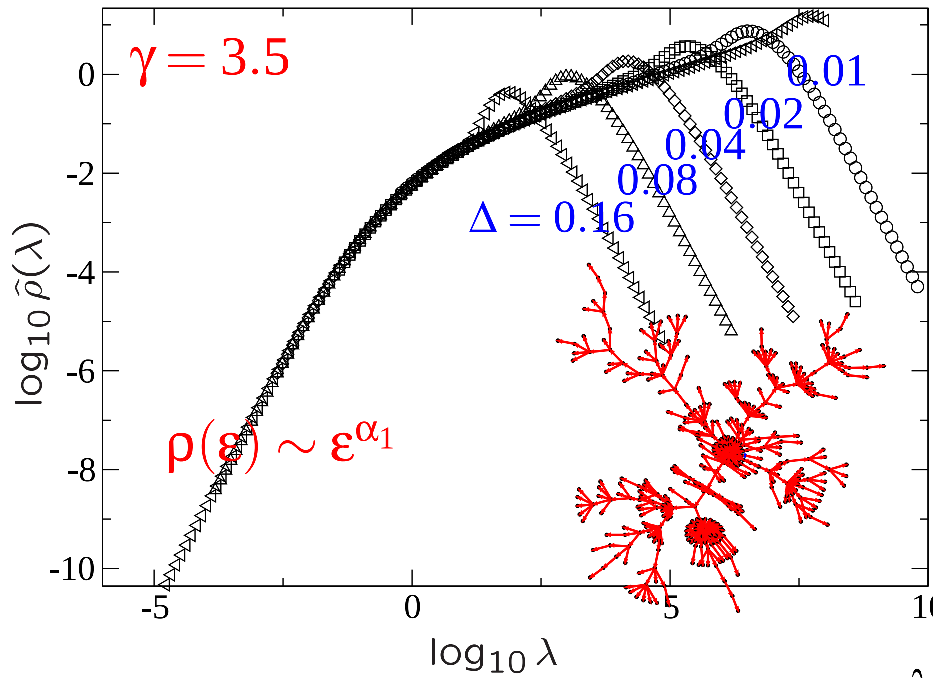




Spectra for scalefree graphs

degree distribution $p(k) \sim k^{-\gamma}$





$p_k \sim k^{-\lambda}$

- data collapse for scaling forms
- numerical confirmation of exponent $\alpha_1 = 2\gamma - 5$
- no algebraic behavior for $\gamma \geq 4$: $\rho(\varepsilon) \sim \exp(-\Delta^{3/2}/\varepsilon^{1/2})$
  (diluted Cayley tree) *Jasch, vF, Blumen. PRE 2004.*