Fakultät für Physik und Geowissenschaften
der Universität Leipzig
Linnéstr. 5, D 04103 Leipzig, Tel. (0341)97-32400/ -32653

Fakultätskolloquium

gemeinsam mit dem

Naturwissenschaftlich-Theoretischen Zentrum

im Rahmen des NTZ-Workshops „CompPhys06“

Am Donnerstag, dem 30. November 2006, 17:15 Uhr
spricht im Kleinen Hörsaal der Fakultät, Linnéstr.5, 04103 Leipzig

Herr Prof. Dr. Volker Dohm
Institut für Theoretische Physik B der RTWH Aachen

über

Universality and diversity of critical phenomena

Alle Interessenten sind herzlich eingeladen.

Prof. Dr. T. Butz	*Prof. Dr. W. Janke*
Dekan	*Direktor des NTZ*

Critical phenomena are divided into so-called "universality classes" characterized only by the dimension d of the system and the number n of components of the order parameter. A prominent example is the (d = 3, n = 2) universality class of the superfluid transition of ^{4}He and of XY spin models. Several experiments under microgravity conditions supported by NASA are discussed that are designed to test the universality predictions of the renormalization-group theory. Another important universality class is the (d = 3, n = 1) case: ordinary fluids and uniaxial magnets. Within this universality class, critical phenomena have recently been predicted to exhibit a considerable diversity due to van der Waals interactions and anisotropy effects. The latter violate the so-called "two-scale factor universality". Analytic predictions of nonuniversal anisotropy effects near T_c are presented that can be tested by Monte Carlo simulations for anisotropic Ising models.

Alle Besucher und Gäste dieses Fakultätskolloquiums sind hiermit zu einer
Gesprächs-Kaffee-Runde mit Herrn Prof. Dr. V. Dohm
während der Poster-Session und Kaffeepause des NTZ-Workshops CompPhys06
(http://www.physik.uni-leipzig.de/~janke/CompPhys06/)
***in der Aula des Fakultätsgebäudes** Linnestr. 5 herzlich eingeladen.*

Universality and diversity of critical phenomena

V. Dohm

Aachen University

I. Universality of critical behavior

- Universality classes

II. n = 2 universality class : ^{4}He and XY models

- experiments in microgravity
- RG theory and MC simulations

III. Two - scale - factor universality

IV. Diversity of critical behavior

- Finite systems
- Van der Waals interaction
- Anisotropy

V. Theoretical predictions

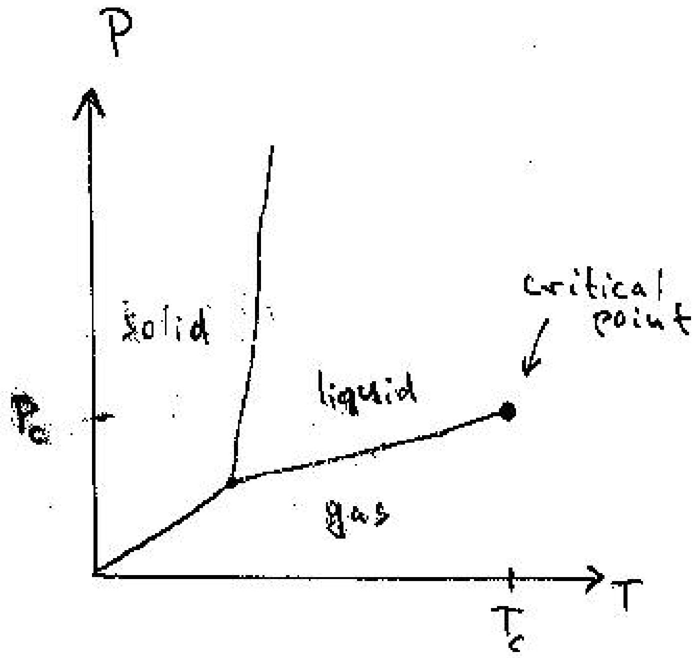
- Binder cumulant, free energy
- Non-rectangular geometries
- Challenge to MC simulations

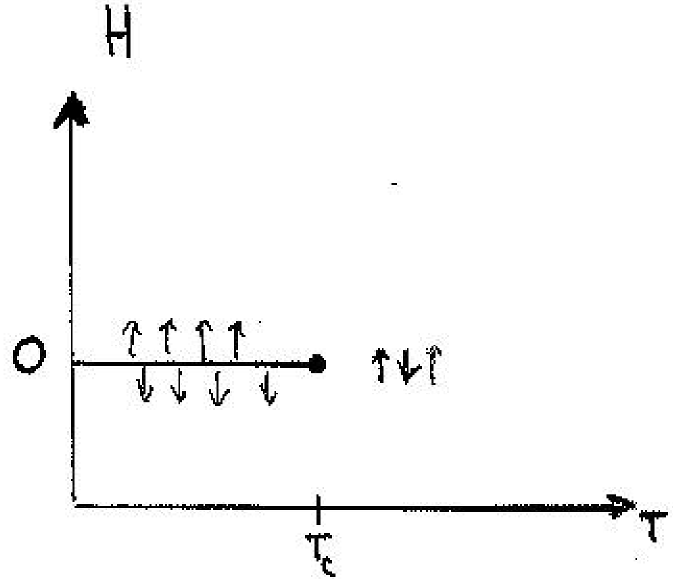
7th NTZ-Workshop on Computational Physics, Leipzig, 30.11.06

Analogy

Liquid / gas

Ferromagnet

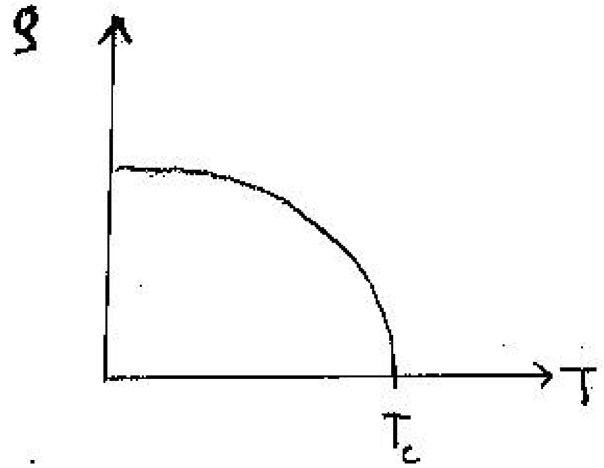


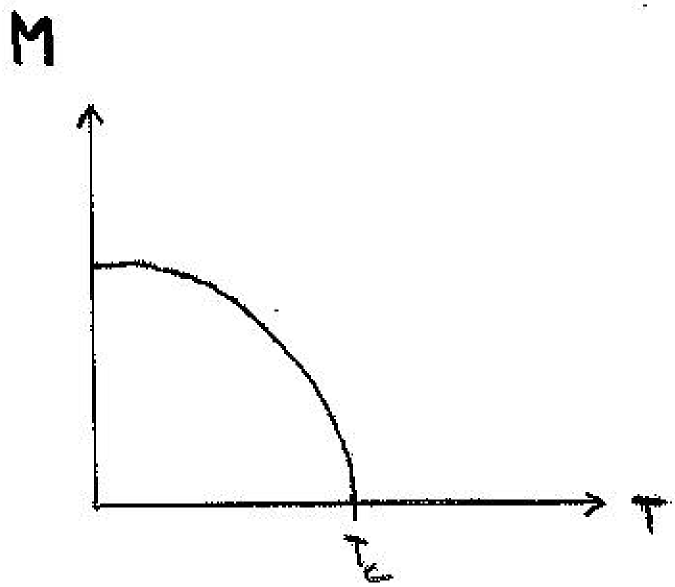


Density difference

$\rho_{liquid} - \rho_{gas}$

Magnetization at $H = O$





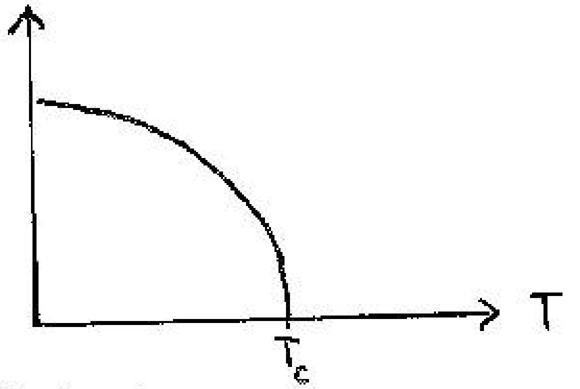
Critical point of fluids

corresponds to Curie-Point of ferromagnets

Singular (critical) behavior :

<u>POWER LAWS</u>

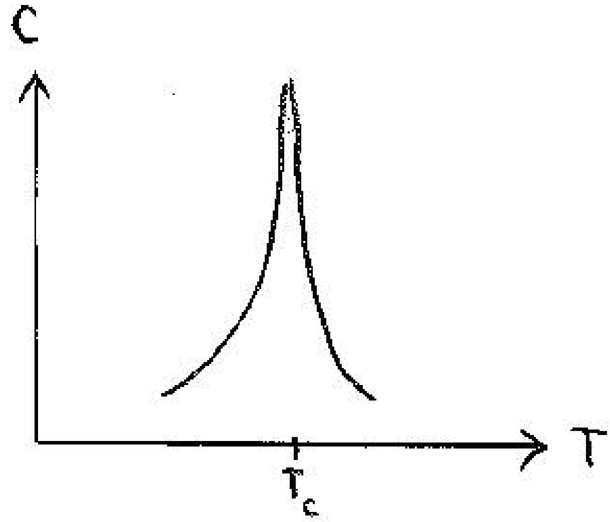
Order-parameter (density diff. , magnetization)
superfluid density



$$\sim (T_c - T)^{\beta}$$

$$\sim (T_c - T)^{\nu}$$

Specific heat



$$\sim |T_c - T|^{-\alpha}$$

Theoretical $\boxed{\text{RG}}$ prediction : $\alpha, \beta, \ldots, \nu$ **UNIVERSAL**

"Renormalization Group" Theory

" "

independent of material properties and of details of interaction :

$\boxed{\text{Fundamental constants of nature}}$

Same for fluids and solids

Amplitudes :

$$C(T) = A^{\pm} \, |t|^{-\alpha} \quad , \quad t = \frac{T - T_c}{T_c}$$

$$\frac{A^{+}}{A^{-}} = \text{universal amplitude ratio}$$

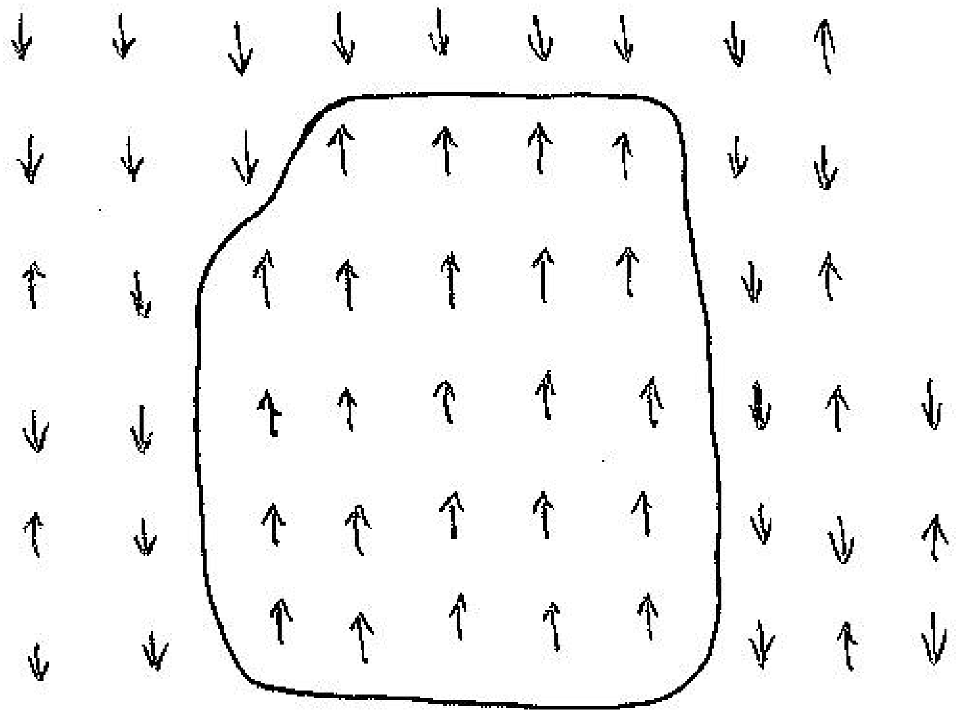
Scaling function :

$$C(T, h) = |t|^{-\alpha} \, f_{\pm}\left(h \, |t|^{-\Delta} \right)$$

$$f_{\pm}(y) = \text{universal scaling function}$$

Same for fluids and solids

Origin of singularities and of universality :



$T \gtrsim T_c$

long-range correlations near T_c

between flu ctuations of order parameter.

Dominant length scale of ordered clusters :

" correlation length " ξ

insensitive to microscopic details :

universality : $\xi \longrightarrow \infty$

$\xi \sim |T - T_c|^{-\nu}$

Universality class	Order parameter	System
$n = 1$ scalar order parameter	Magnetization M_Z (parallel to anisotropy axis) density difference $\rho_F - \rho_G$	Anisotropic magnets Ising models All fluids
$n = 2$ two component order parameter	Magnetization $M = \begin{pmatrix} M_x \\ M_y \end{pmatrix}$ (perpendicular to anisotropy axis) complex wave function (Bose condens) ← $\psi = \psi' + i\psi''$	Planar magnets (rotational symmetry) XY models Superfluid Helium 4 (Superconductors)

Within a universality class: <u>same critical exponents and scaling function</u>

Experimental test of

<u>prediction of universality</u>

is

<u>test of a fundamental theory</u>

of physics :

“Renormalization Group theory“

great importance in many

areas of physics :

Statistical physics

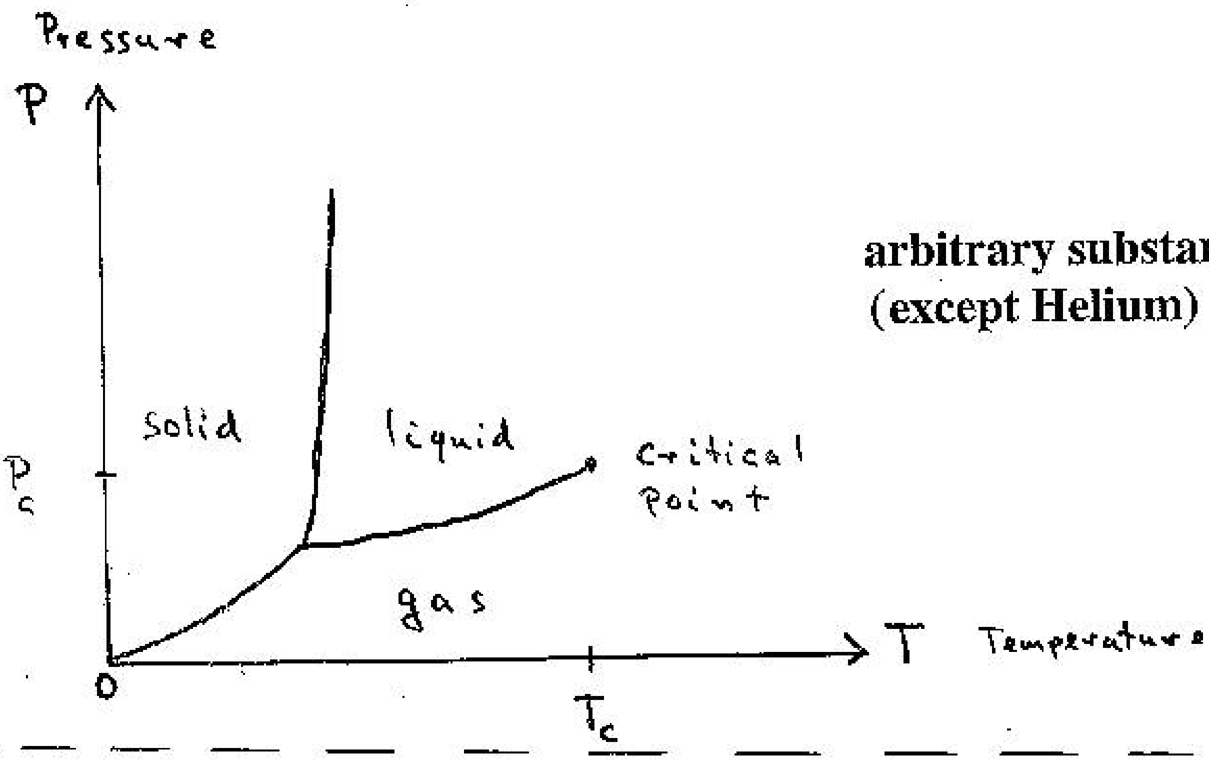
Elementary particle physics

Hydrodynamics (Turbulence, ...)

Solid state physics (electronic properties, ...)

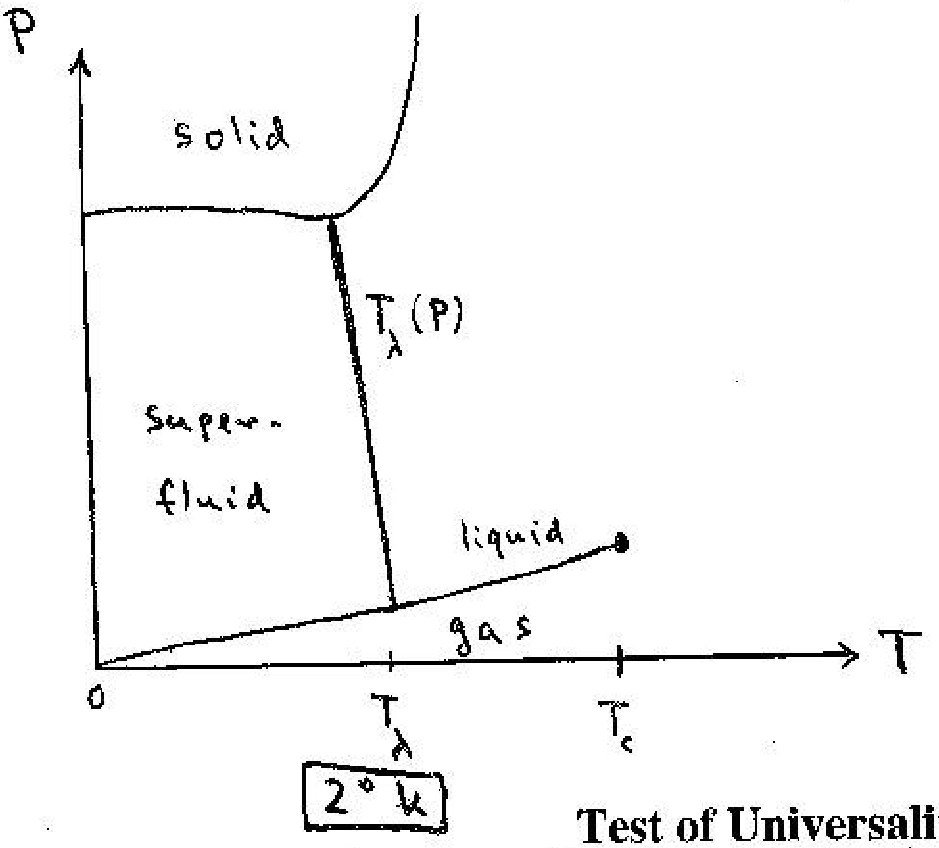
∞ many degrees of freedom coupled

Phase diagram and critical points



arbitrary substance
(except Helium)

Helium



Line of critical points

$T_\lambda(P)$

small thermal noise

extremely pure

$|T - T_\lambda| \approx 10^{-10}$ K

Test of Universality

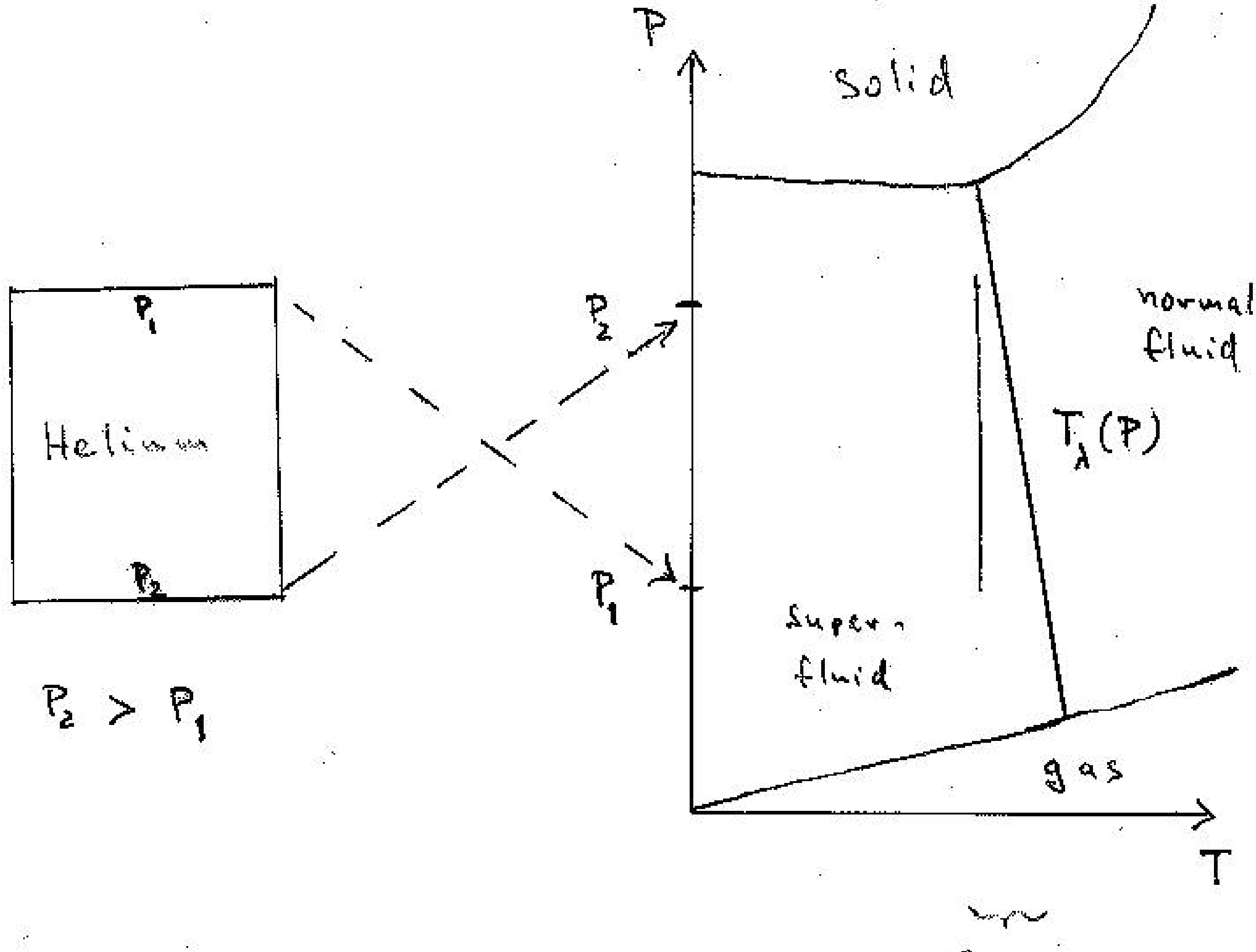
Measurements at various pressures P

Question : Critical exponents independent of P ?

Experiment on earth:

Problem due to gravity

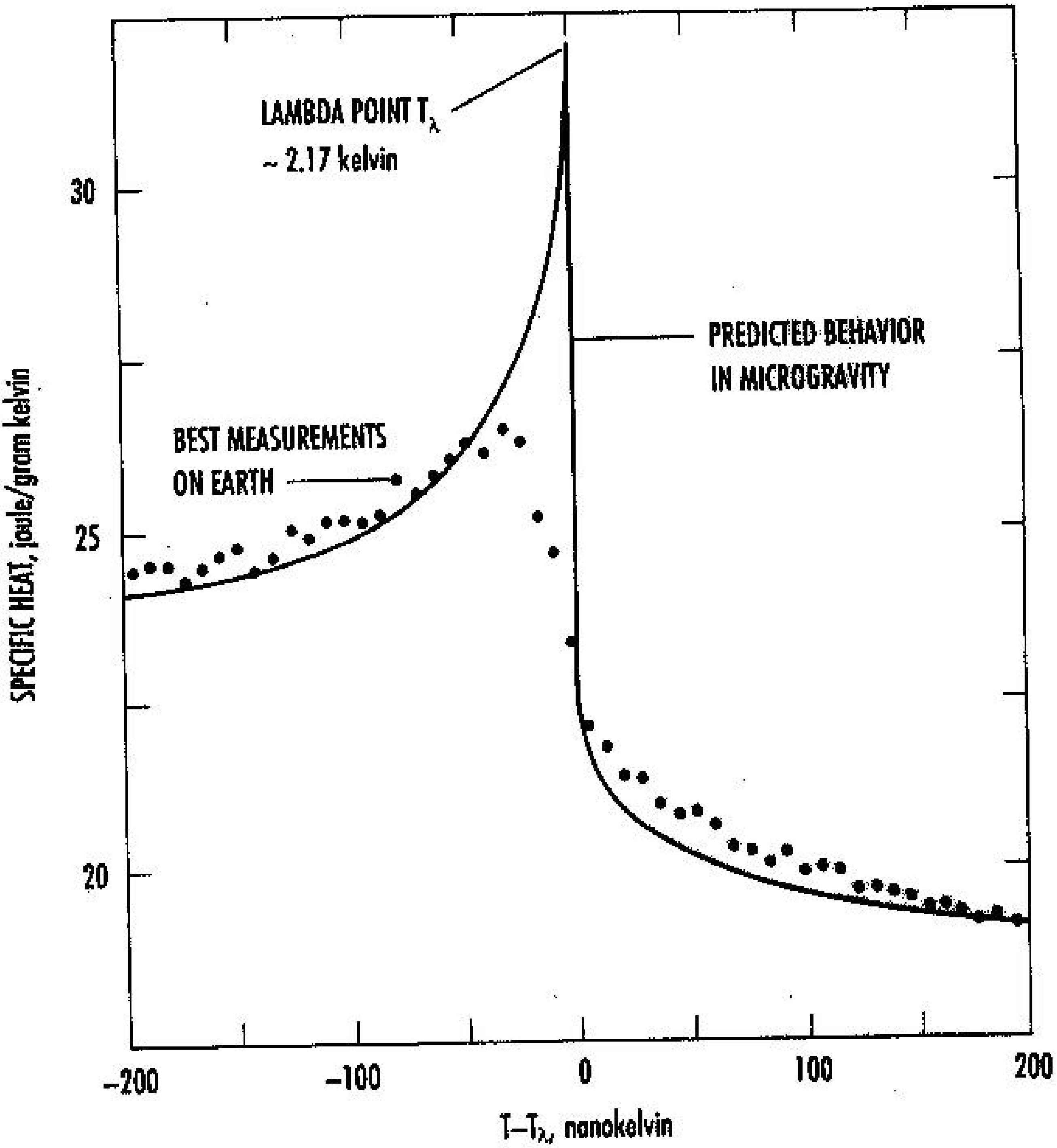
gravity induces pressure gradient



Uncertainty of temperature distance from T_λ

⇒ inaccuracy of measurements

⇒ gravitationally induced rounding

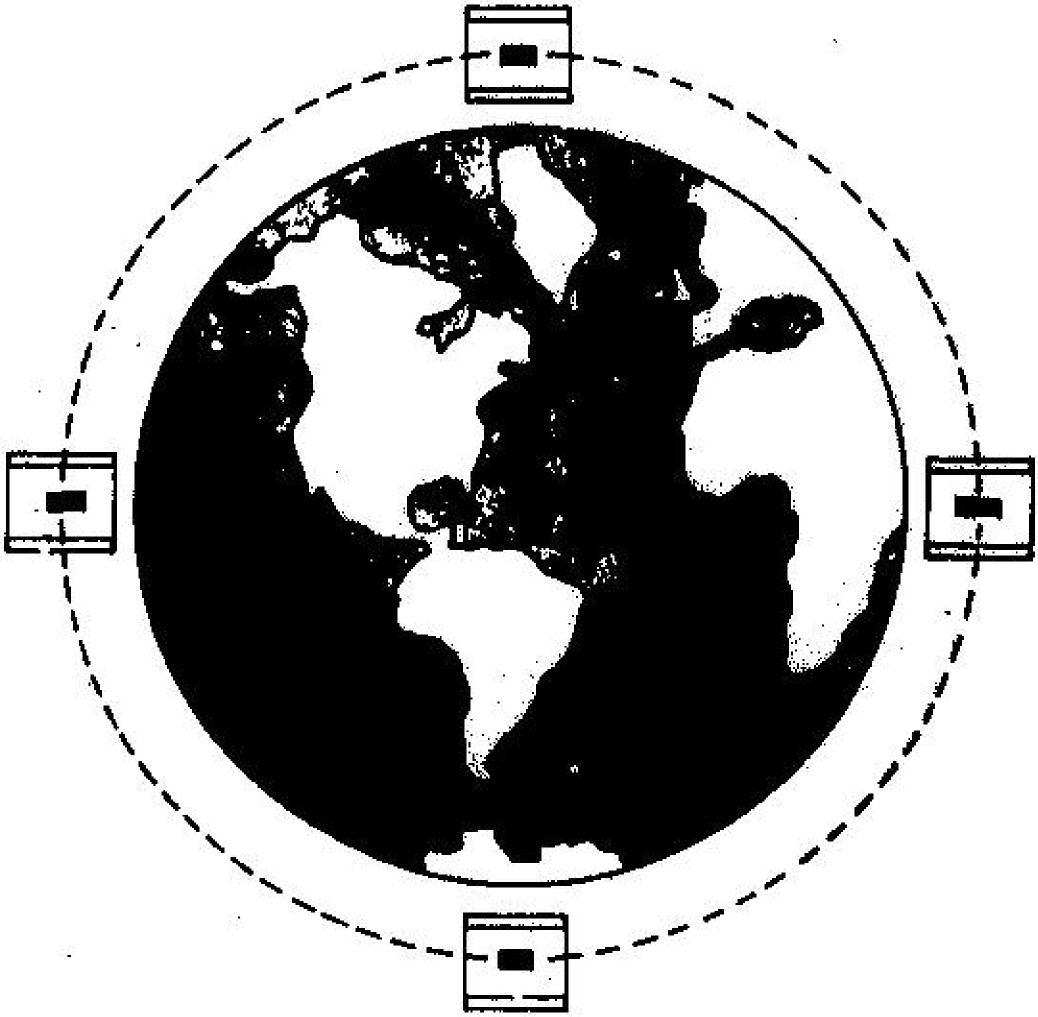


$$C = A^{\pm} \cdot \left| \frac{T - T_\lambda}{T_\lambda} \right|^{-\alpha}$$

Space shuttle or ISS in earth orbit

compensation of gravity (g)

by centrifugal force

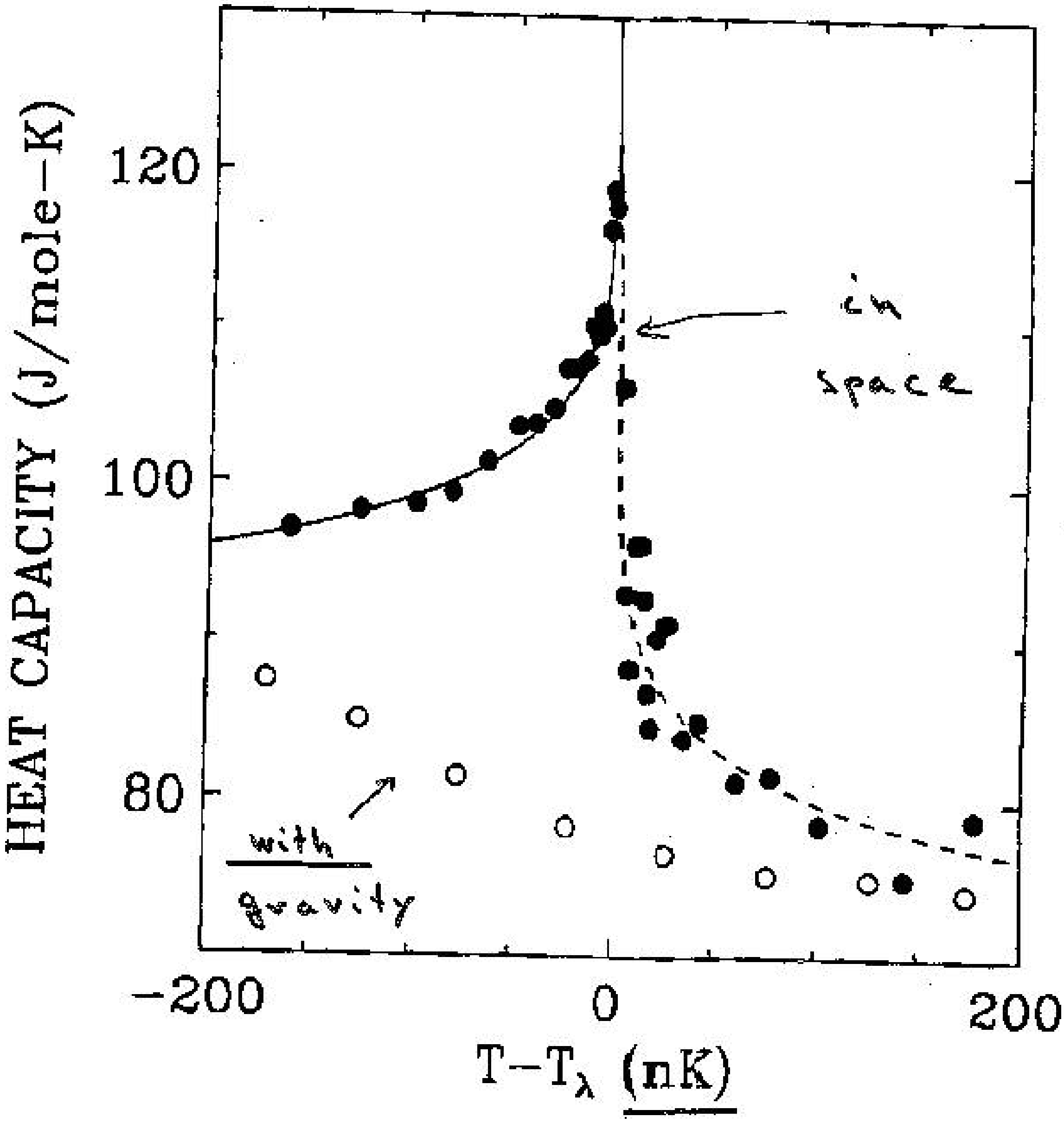


microgravity (μ g)

Bulk specific heat near T_λ

[Lipa et al., Phys. Rev. Lett. 76, 944 (1996)]

Bulk specific heat near T_λ



Columbia USMP-1 1992

$$C = A^{\pm} \, |T - T_\lambda|^{-\alpha} + B$$

$\alpha = -0.0127 \pm 0.0003$, $\frac{A^+}{A^-} = 1.053 \pm 0.002$

Most accurate experim. result in cond. mat. p

Test of RG theory

and of universality :

Compare experimental result with

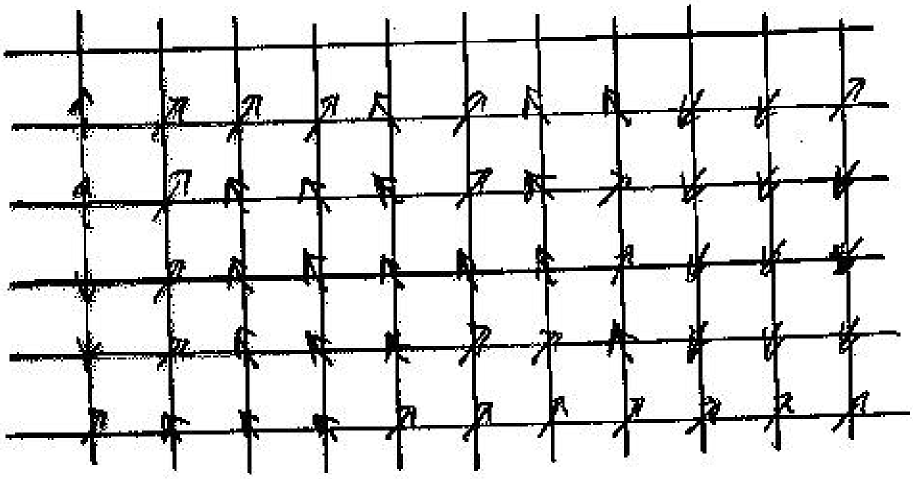
α , $\dfrac{A^+}{A^-}$ of lattice model

and

α , $\dfrac{A^+}{A^-}$ of φ^4 field theory

$n = 2$ universality class

XY lattice model



$$H = \sum_{\langle ij \rangle} J \, \vec{S}_i \cdot \vec{S}_j$$

2 - component spins $\vec{S}_i = \begin{pmatrix} S_{x\,i} \\ S_{y\,i} \end{pmatrix}$

on lattice points i

$$|\vec{S}_i| = 1$$

Exchange interactions J

between nearest neighbors

Monte Carlo simulations

($n = 1$: Ising model : $S_i = \pm 1$)

Continuum and lattice Hamiltonian of φ^4 theory

fluctuating n-component vector field $\varphi(\mathbf{x})$ in finite volume $V = L^d$:

$$H_{field} = \int_V d^d x \left[\frac{r_0}{2}\varphi^2 + \frac{1}{2}(\nabla\varphi)^2 + u_0(\varphi^2)^2\right]$$

$n = 2$: $\varphi = \begin{pmatrix} \varphi^{(} \\ \varphi^{(} \end{pmatrix}$

fluctuating n-component vector variables φ_i on lattice points $\mathbf{x}_i$

$$H_{lattice} = \sum_i \left[\frac{r_0}{2}\varphi_i^2 + u_0(\varphi_i^2)^2\right] + \sum_{i,j}\frac{1}{2}J_{ij}(\varphi_i - \varphi_j)^2$$

temperature variable : $\quad r_0 = r_{0c} + b_0\, t\,, \quad t = (T - T_c)/T_c$

Universality class : dimension d , number n of components of vector φ

RG theory :

universal critical behavior for $t \to 0$:

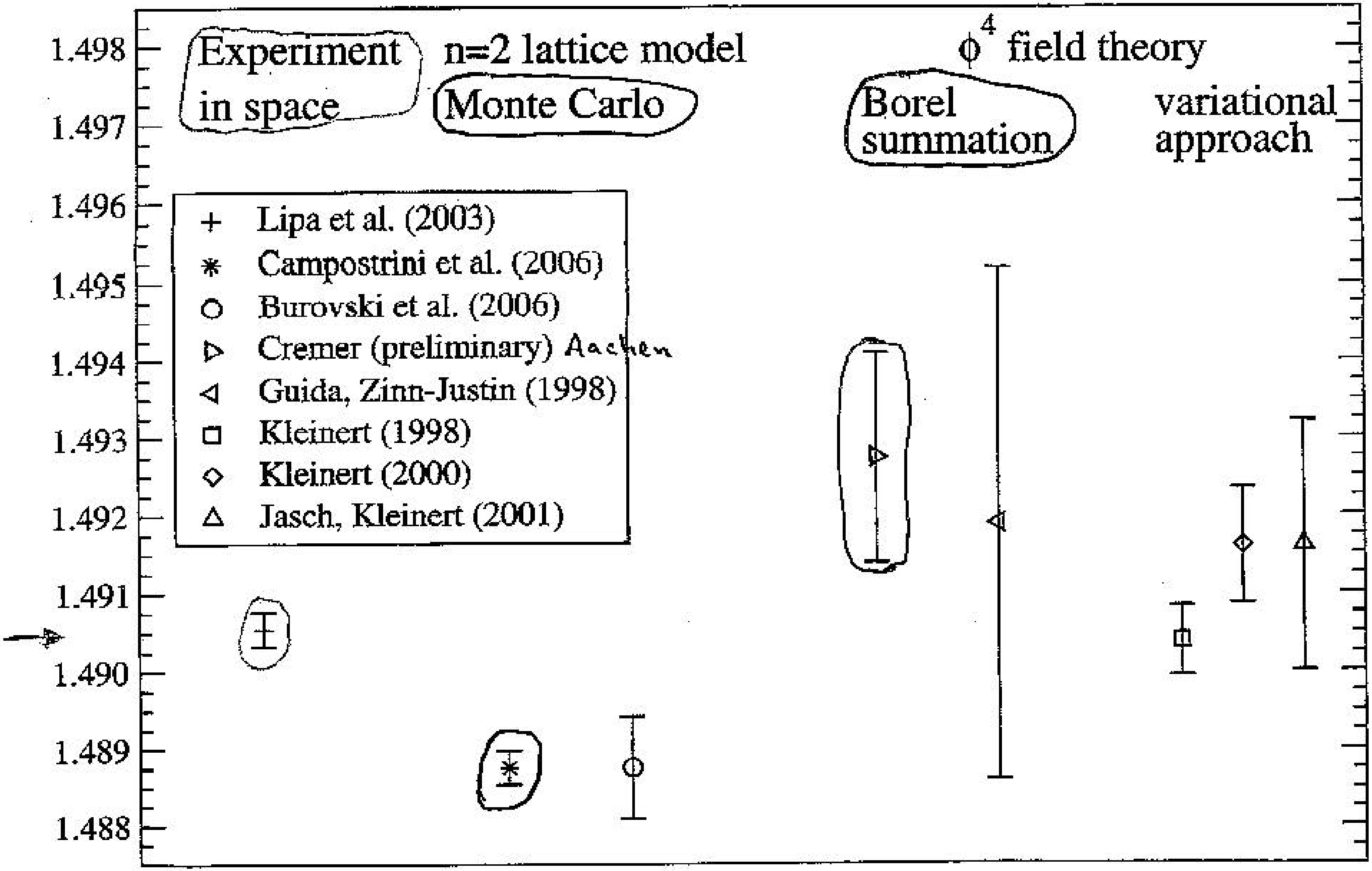
all critical exponents and scaling functions are universal $\to$ the same for given d, n,

independent of **nonuniversal parameters** : b_0 , u_0 , Λ (cutoff) , J_{ij} ,

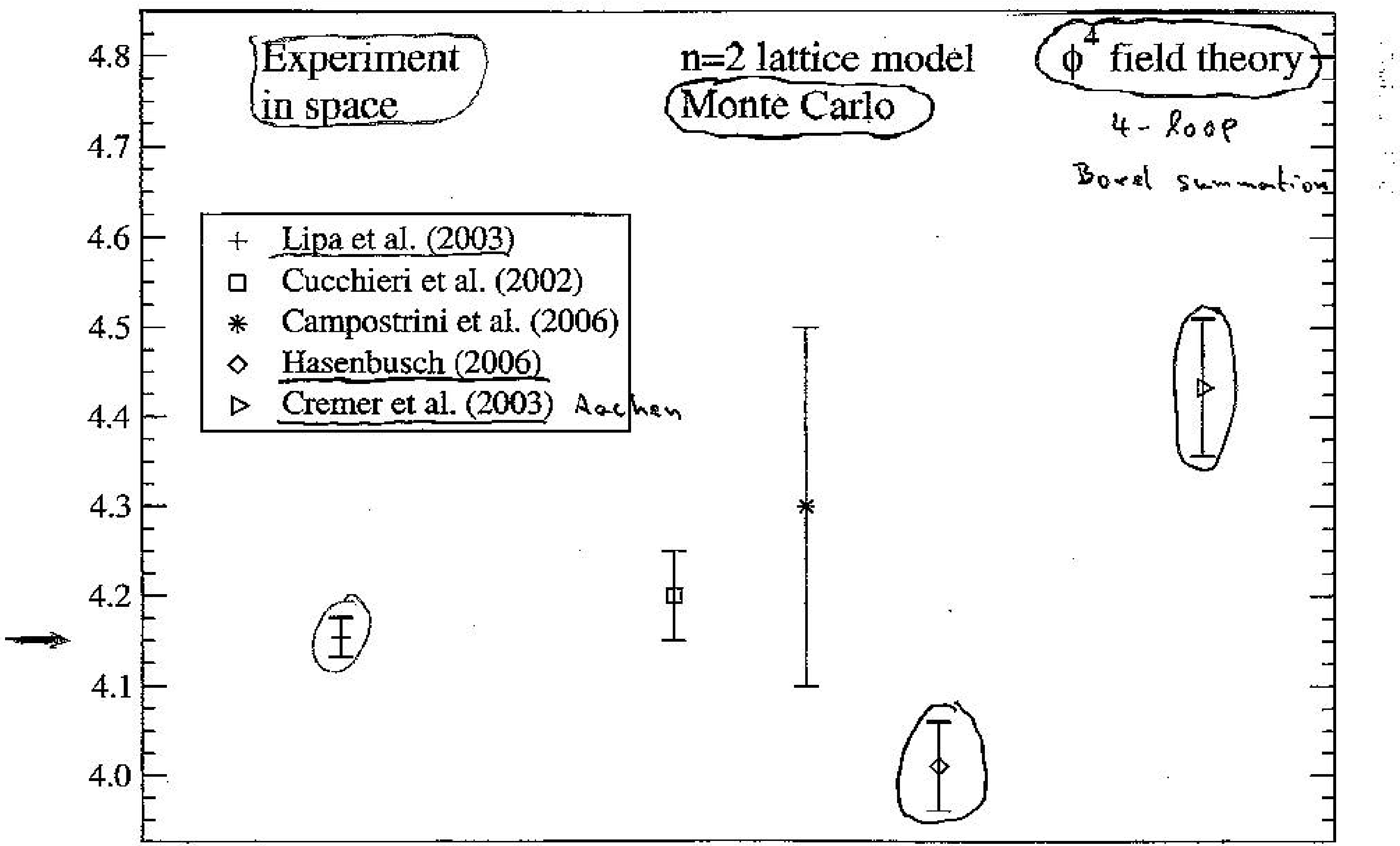
independent of lattice structure

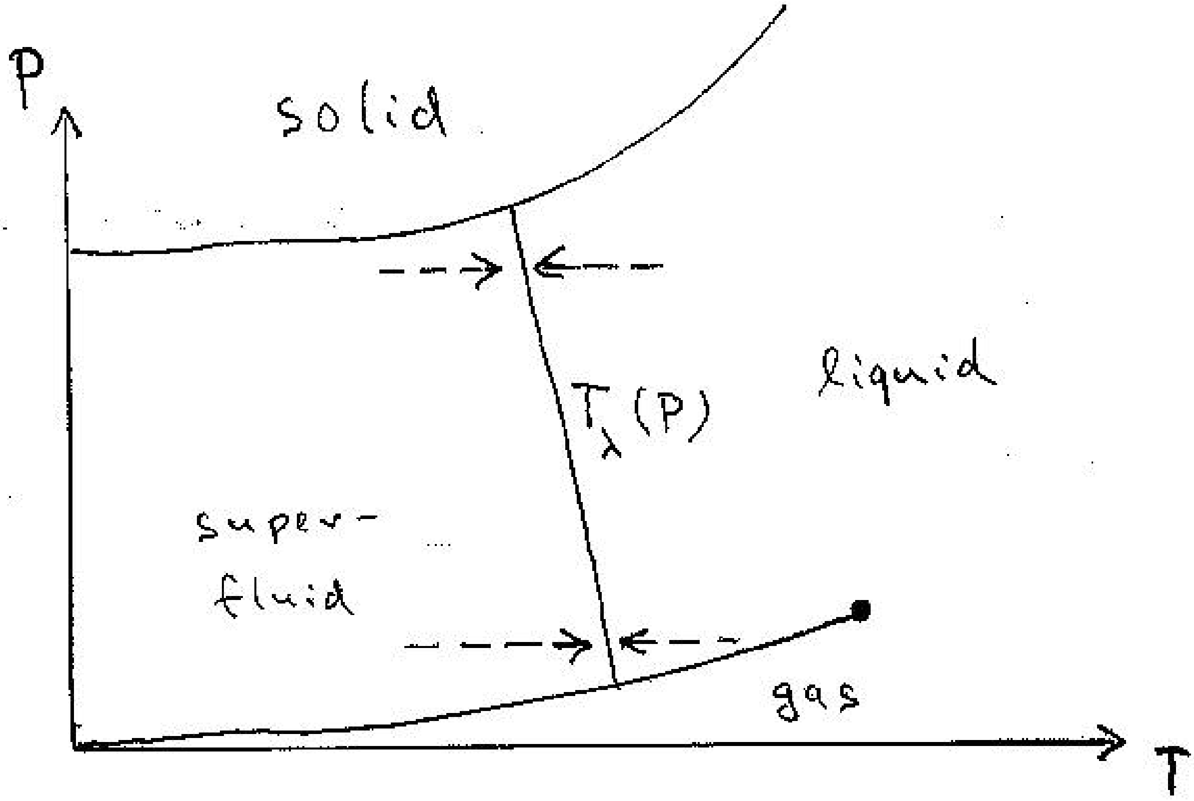
Critical exponent 1/ν

$\nu = \frac{1}{3}(2-\alpha)$ exact



Universal amplitude combination P $= \frac{1}{\alpha}\left(1 - \frac{A^+}{A^-}\right)$





Test of Universality

Superfluid density: $\rho_s = A_\rho \, |T - T_\lambda|^\nu$

Specific heat: $C = A^{\pm} \, |T - T_\lambda|^{-\alpha}$

for several pressures P along the λ-line of ^{4}He
and several concentrations c_3 of ^{3}He-^{4}He-mixtures

universal critical exponents ν , α ,

universal amplitude ratios $\dfrac{A^+}{A^-}$, $\dfrac{A_\rho}{(A^-)^{1/3}}$

independent of P and c_3 ?

exact relation $2 - \alpha = 3\nu$?

(19)

REVIEWS OF MODERN PHYSICS, VOLUME 78, JULY–SEPTEMBER 2006

Critical phenomena in microgravity: Past, present, and future

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This review provides an overview of the progress in using the low-gravity environment of space to explore critical phenomena and test modern theoretical predictions. Gravity-induced variations in the hydrostatic pressure and the resulting density gradients adversely affect ground-based measurements near fluid critical points. Performing measurements in a low-gravity environment can significantly reduce these difficulties. A number of significant experiments have been performed in low-Earth orbit. Experiments near the lambda transition in liquid helium explored the regime of large correlation lengths and tested the theoretical predictions to a level of precision that could not be obtained on Earth. Other studies have validated theoretical predictions for the divergence in the viscosity as well as the unexpected critical speeding up of the thermal equilibrium process in pure fluids near the liquid-gas critical point. The scientific content of previously flown low-gravity investigations of critical phenomena as well as those in the development stage, and associated ground-based work are described.

General Relativity and Gravitation, Vol. 36, No. 3, March 2004 (© 2004)

REVIEW

Experiments in Fundamental Physics Scheduled and in Development for the ISS

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Received September 19, 2003; revised October 6, 2003

This is a review of those experiments in the area of Fundamental Physics that are either approved by ESA and NASA, or are currently under development, which are to be performed in the microgravity environment of the International Space Station. These experiments cover the physics of liquid Helium (SUE, BEST, MISTE, DYNAMX, and EXACT), ultrastable atomic clocks (PHARAO, PARCS, RACE), ultrastable microwave resonators (SUMO), and particle detectors (AMS and EUSO). The scientific goals are to study more precisely the universality properties of liquid Helium under microgravity conditions, to establish better time standards and to test the universality of the gravitational red shift, to make more precise tests of the constancy of the speed of light, and to

Fundamental question :

How many independent nonuniversal amplitudes exist within a (d, n) bulk universality class ?

Partition function of φ^4 theory

$$Z_{field} = \int \mathcal{D}\varphi \ \exp(-H_{field}/k_B T)$$

$$Z_{lattice} = \int \mathcal{D}\varphi \ \exp(-H_{lattice}/k_B T)$$

Free energy per unit volume near criticality

$$f(t,h) = -\frac{k_B T}{V} \ln Z = f_{sing}(t,h) + f_{reg}(t,h)$$

$$t = \frac{T - T_c}{T_c}$$

Standard asymptotic scaling form near T_c $(t \ll 1)$, small h :

$$f_{sing}(t,h) = A_1 \, t^{d \cdot \nu} \cdot W(A_2 h \, t^{-\beta\delta})$$

Universal scaling function W, universal critical exponents ν, $\beta\delta$

Two nonuniversal bulk amplitudes A_1 , A_2

Bulk order-parameter correlation function for $T \geq T_c$, $h = 0$

$$G_{field}(t, \mathbf{x}) = < \varphi(\mathbf{x})\varphi(0) > , \qquad G_{lattice}(t, \mathbf{x_i} - \mathbf{x_j}) = < \varphi_i \, \varphi_j >$$

Second-moment bulk correlation length

$$\xi_{field} = \left[\frac{1}{2d} \, \frac{\int d^d x \, \mathbf{x}^2 \, G_{field}(t, \mathbf{x})}{\int d^d x \, G_{field}(t, \mathbf{x})} \right]^{1/2} , \qquad \xi_{lattice} = \left[\frac{1}{2d} \, \frac{\sum_{\mathbf{x}} \mathbf{x}^2 \, G_{lattice}(t, \mathbf{x})}{\sum_{\mathbf{x}} G_{lattice}(t, \mathbf{x})} \right]^{1/2}$$

standard asymptotic bulk scaling form near T_c $(t \ll 1)$ for large $|\mathbf{x}| \gg \Lambda^{-1}, \tilde{a}$:

$$G_{scal}(t, \mathbf{x}) = A_G \, |\mathbf{x}|^{-d+2-\eta} \, \Phi(\mathbf{x}/\xi) ,$$

$$\xi = \xi_0 \, t^{-\nu} ,$$

universal scaling function Φ, universal critical exponents η and ν,

$$\boxed{\text{two nonuniversal bulk amplitudes } A_G \text{ and } \xi_0 .}$$

Fundamental question :

How many independent nonuniversal amplitudes exist within a (d, n) bulk universality class ?

Bulk free energy density :

$$f_{sing}(t,h) = A_1 t^{d\cdot\nu} \cdot W(A_2 h\, t^{-\beta\delta})$$

Bulk correlation function : $(h=0)$

$$G(r,t) = A_G \frac{1}{r^{d-2+\eta}} \Phi\,(r/\xi)$$

correlation length $\xi = \xi_0\ t^{-\nu}$

4 amplitudes

Sum rule :

$$\int d^d r\ G(r,t) = \chi = \frac{\partial^2 f(t,h)}{\partial h^2}\Big|_{h=0}$$

→ Relation between A_G, ξ_0 , A_1, A_2 → 3 independent amplitudes

---

RG theory : $f_{sing}(t,0)\cdot\xi^d = Q(d,n) =$ universal

$$\xi_0 = (Q/A_1)^{1/d} \quad : \text{not independent}$$

⇒ $\boxed{\text{only 2 independent amplitudes}}$

"2-scale-factor universality" for infinite systems

[Stauffer et al.(1972), Hohenberg et al. (1976), Wegner(1976)]

Widely accepted statements

1. RG theory proves the validity of

universality of critical phenomena

in bulk and confined systems

2. Within a universality class, all systems have the same critical exponents and scaling functions

3. Only two nonuniversal amplitudes: **2-scale factor universality**

Main message of this talk :

Statements not true in general.
RG theory is compatible with a certain

diversity of critical phenomena

within a given universality class.

Origin of diversity

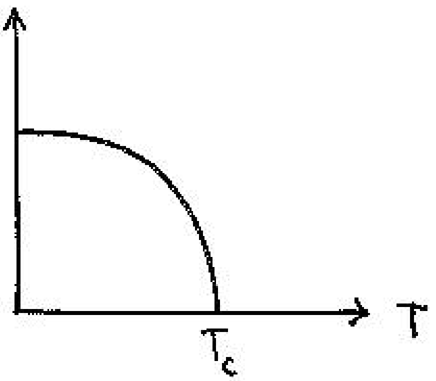
- finite-size effects
- van der Waals interaction in fluids
- anisotropy in solids

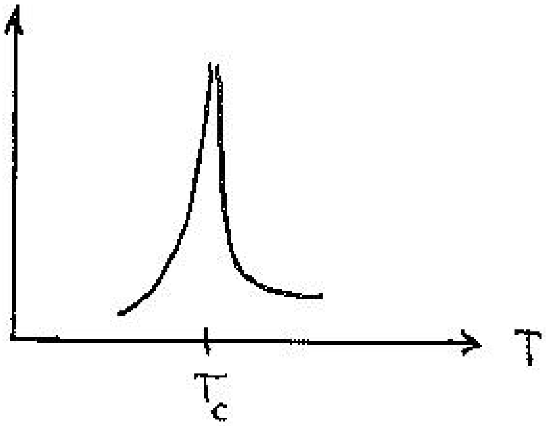
Critical exponents remain universal,

but

- finite-size scaling functions
  depend on geometry, boundary conditions
- van der Waals interaction destroys scaling and universalit
  in the regimes
- anisotropy makes finite-size scaling functions
  nonuniversal even at T_c,

ideal systems: infinite, homogeneous



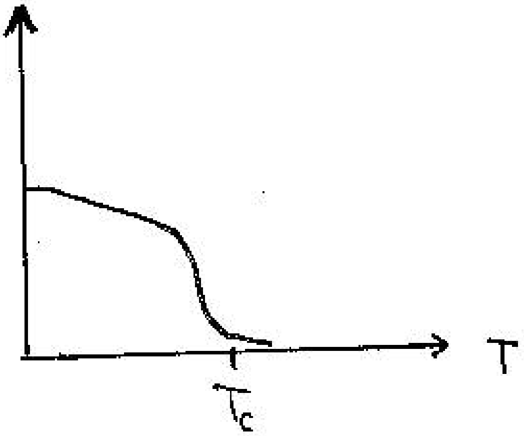


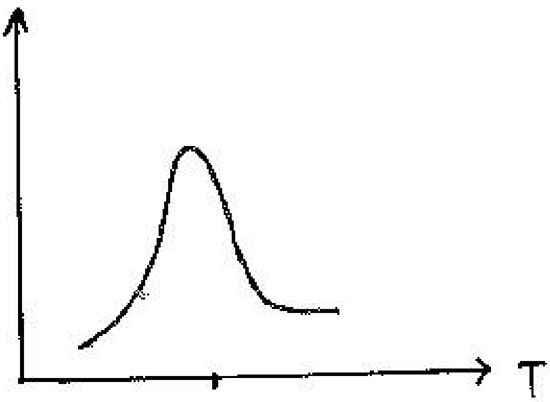
Divergences, singularities

Universality (d, n)

$\left.\begin{array}{l}\textbf{Real systems}\\\textbf{Computer models}\end{array}\right\}$ finite, with surfaces

$\Rightarrow$ Singularities rounded





dependent on geometry, boundary conditions

<u>Hypothesis :</u>

Two-scale factor universality valid also for **confined** systems

[V. Privman, M.E. Fisher, Phys. Rev. B **30**, 322 (1984)]

$$f(t,h,L) = f_{sing}(t,h,L) + f_{reg}(t,L) \quad , \quad t = \frac{T - T_c}{T_c}$$

$$f_{sing}(t,h,L) = L^{-d}\, Y(C_1 t L^{1/\nu}\, ,\, C_2 h L^{\beta\delta/\nu})$$

"universal finite-size scaling"

$Y(x,y)$ **universal** for all systems within a (d,n) **universality class** for given geometry and boundary conditions

Long-range interactions :

$$U(\mathbf{x}-\mathbf{x}') \quad \sim \quad \frac{\mathbf{b}}{|\mathbf{x}-\mathbf{x}'|^{\mathbf{d}+\sigma}}$$

real fluids : $d = 3$, $\sigma = 3$

van der Waals

φ^4 theory :

$$H \;=\; \int_{\mathbf{k}} \frac{1}{2}\Big(r_0 + k^2 + b\,k^\sigma\Big)\hat{\varphi}_k\hat{\varphi}_{-k} \quad + \quad u_0 \int d^d x\; \varphi^4$$

Order-parameter correlation function :

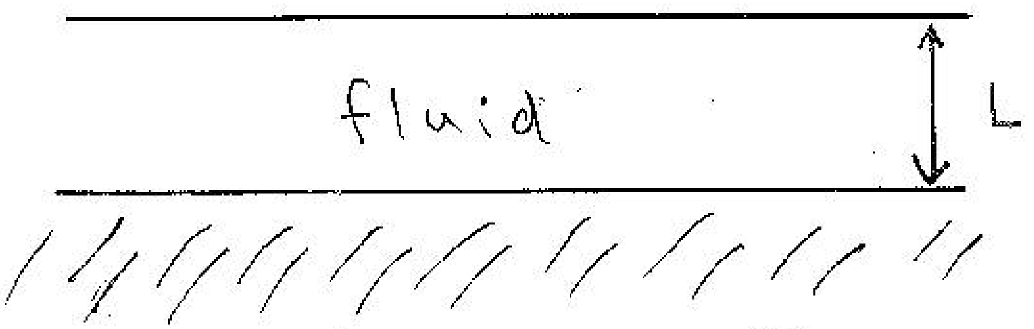
$$<\varphi(r)\;\;\varphi(0)> \sim \frac{A}{r^{d-2+\eta}}\,\Phi\left(\frac{r}{\xi}\right) \quad + \quad \frac{c\;\;\xi^{4-2\eta}}{r^{d+\sigma}}$$

scaling non-scaling

η, Φ universal non-universal

also : finite-size effects non universal

fluid film of thickness L
L dependent free energy



$$t = \frac{T - T_c}{T_c}$$ reduced temperature

"Casimir force" (per unit area)

$$K(t, L) = - \frac{\partial}{\partial L} f_{ex}(t, L)$$

excess free energy (per unit area)

$$f_{ex}(t, L) = f_{film}(t, L) - L \cdot f_{bulk}(t)$$

<u>singular</u> and regular parts:

$$K(t, L) = K_{sing}(t, L) + K_{reg}(t, L)$$

<u>experiments</u>: R. Garcia and M.H.W. Chan,
in ^4He PRL <u>83</u>, 1187 (1999); <u>88</u>, 086101 (2002)

Previous theory for fluids and ^{4}He

scaling form:

$$K_{sing}(t, L) = L^{-d} \underline{\vartheta(L/\xi)}$$

universal scaling function (only if isotropic)

$L \gg \xi$:

$$\vartheta(L/\xi) \underset{L \gg \xi}{\sim} c \exp[-L/\xi]$$

universal finite-size scaling

New theory (including long-range interaction ~ b)
[X.S. Chen, V. D., Phys. Rev. E (2002)]

$$K_{sing}(t, L) = L^{-d} \vartheta(L/\xi) + b L^{-d+2-\sigma} B(L/\xi)$$

universal scaling (only if isotropic)

non universal non scaling

$L \gg \xi$:

leading!

$$K_{sing}(t, L) \underset{L \gg \xi}{\sim} c\, e^{-L/\xi} + b\, \xi^2 L^{-d-\sigma}$$

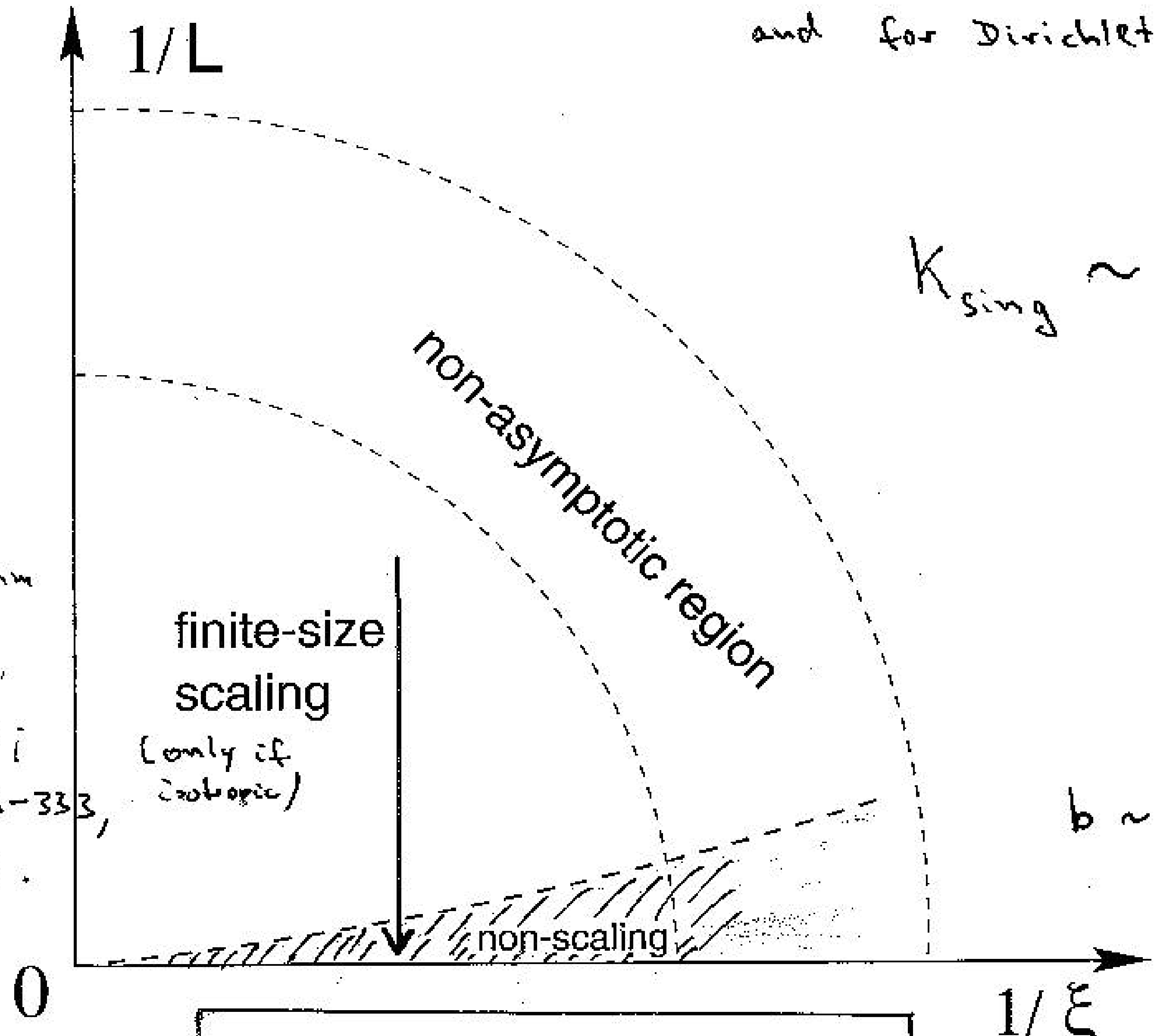
both for periodic b.c.
and for Dirichlet b.c. :

$$K_{sing} \sim b\,\xi^2 L^{-d-\sigma}$$

<u>non-exponential</u>

nonuniversal power law

$b \sim$ strength of long-range interaction

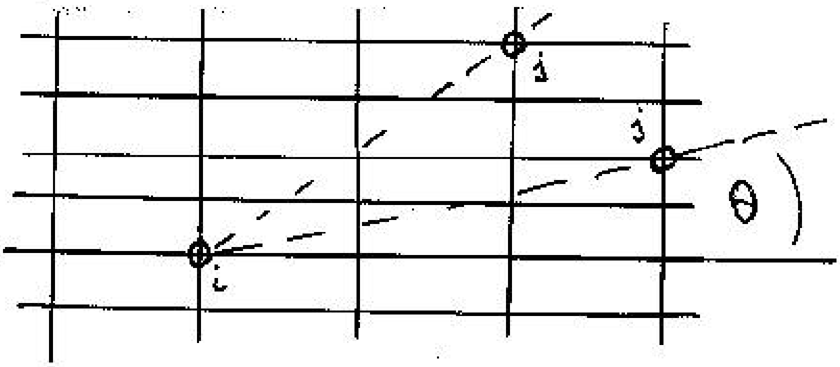


X.S. Chen, V. Dohm
Phys. Rev. E **66**, 16102 (2002);
Physica B 329-333,
202 (2003).

Violation of finite-size scaling

$\neq$ "correction"

Anisotropy in solids (non-cubic symmetry)



Spins S_i on lattice points $\mathbf{x}_i$,
bulk correlation function

$$< S_i S_j > = G(\mathbf{x}_i - \mathbf{x}_j)$$

depends on direction Θ,
is not isotropic, even at $T = T_c$.

need more nonuniversal parameters
than in isotropic fluids
within the same universality class

Main message : { X.S. Chen, V. Dohm, Phys. Rev. E 70, 056136 ('04)
V. Dohm, J. Phys. A 39, L 259 ('06)

Standard argument
"simple rescaling of lengths eliminates anisotropy"
is not correct.

Standard **isotropic** φ^4 **field** theory at $h = 0$:

$$H_{field} = \int_V d^dx \left[\frac{r_0}{2}\varphi^2 + \frac{1}{2}(\nabla\varphi)^2 + u_0(\varphi^2)^2\right]$$

Generalization : **anisotropic** φ^4 **field** theory :

$$H_{field} = \int_V d^dx \left[\frac{r_0}{2}\varphi^2 + \frac{1}{2}\sum_{\alpha,\beta}^{d} A_{\alpha\beta}\frac{\partial\varphi}{\partial x_\alpha}\frac{\partial\varphi}{\partial x_\beta} + u_0(\varphi^2)^2\right]$$

$d \times d$ anisotropy matrix $A_{\alpha\beta} = A_{\beta\alpha}$, contains $\underbrace{d(d+1)/2}$ nonuniversal parameters

$d = 3$: 6 parameters !

Origin : anisotropic φ^4 **lattice** theory :

$$H_{lattice} = a^d \left\{\sum_i \left[\frac{r_0}{2}\varphi_i^2 + u_0(\varphi_i^2)^2\right] + \sum_{i,j}\frac{1}{2a^2}J_{ij}(\varphi_i - \varphi_j)^2\right\}$$

may contain an arbitrary number of nonuniversal parameters J_{ij}

second moments : $A_{\alpha\beta} = \frac{1}{N}\sum_{i,j}(x_{i\alpha} - x_{j\alpha})(x_{i\beta} - x_{j\beta})\, J_{ij}$

Main message

Dependence of finite-size effects on $A_{\alpha\beta}$

not only violation of 2-scale-factor universality,

but also

violation of universality

for all finite-size scaling functions.

Illustration : specific examples : $d = 2$

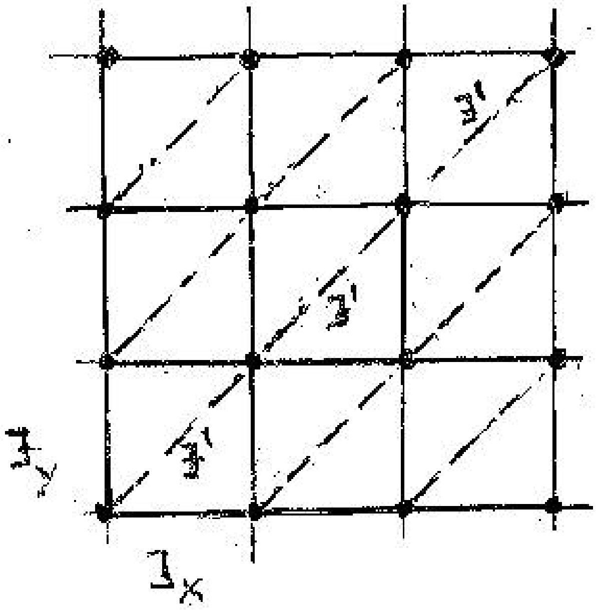
$d = 3$

square lattice, spins on lattice sites

J_x, J_y ferromagnetic nearest-neighbor couplings

J' next-nearest-neighbor coupling

no change of critical exponents!



anisotropic couplings

For $J'=0$: anisotropy can be eliminated by rescaling of lattice constants

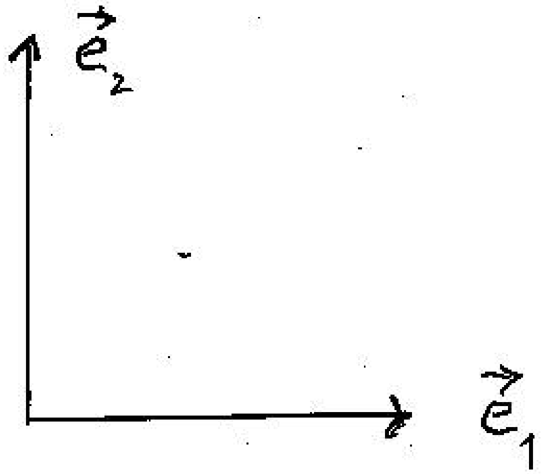
For $J' \neq 0$:

anisotropy matrix $\quad (A_{\alpha\beta}) = 2 \begin{pmatrix} J_x + J' & J' \\ J' & J_y + J' \end{pmatrix}$

$$\det \underline{\underline{A}} > 0 : \quad J' > - \frac{J_x J_y}{J_x + J_y}$$

Eigenvectors $\vec{e}_i$ of $\underline{\underline{A}}$:

$$\underline{\underline{A}}\,\vec{e}_i = \lambda_i\,\vec{e}_i \neq$$ cubic axes (!) in general



" principle axes "

<u>Orientation of</u> $\vec{e}_i$

relative to lattice axes

depends on J'/J_x, J'/J_y , nonuniversal

need $\frac{1}{2}\, d \cdot (d-1)$ angles to specify $\vec{e}_i$

need d eigenvalues λ_i

Must define

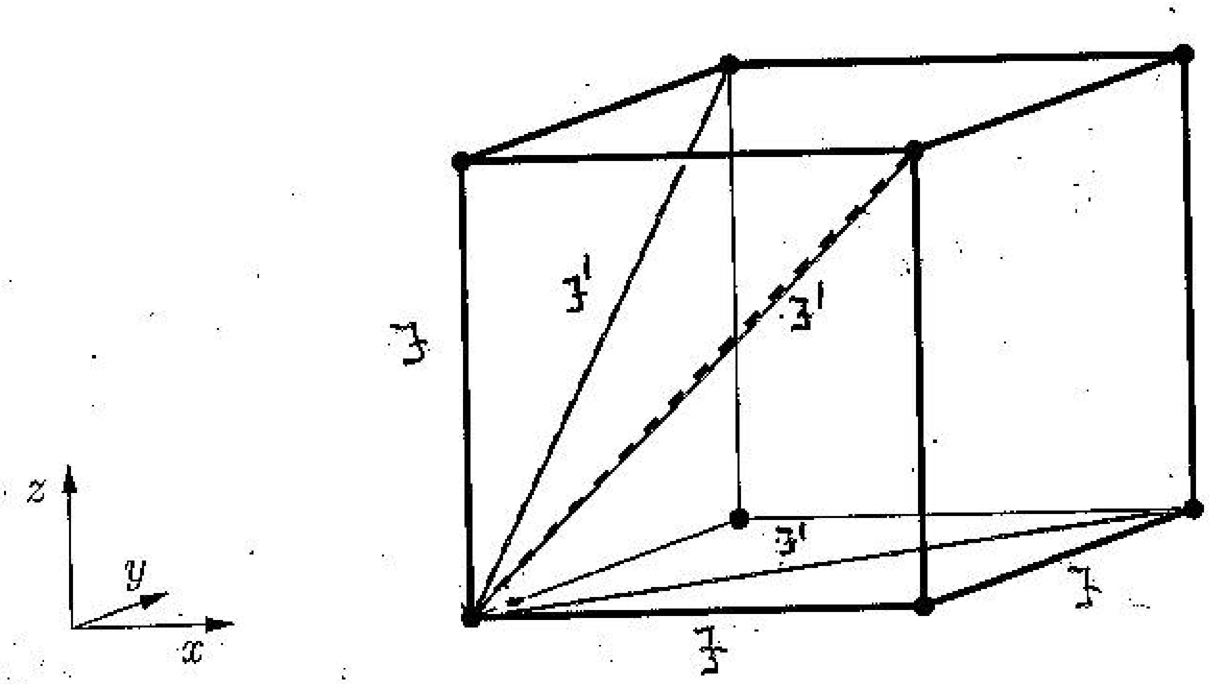
" principle correlation lengths "

$$\xi_i = \xi_i^{(0)}\, t^{-\nu} \qquad i = 1, \ldots, d$$

nonuniversal amplitudes depend on direction i

$$\xi_i^{(0)} / \xi_j^{(0)} = \left(\lambda_i / \lambda_j \right)^{1/2}$$

d = 3



along cubic axes
nearest-neighbor coupling $\quad J \quad$ "isotropic"

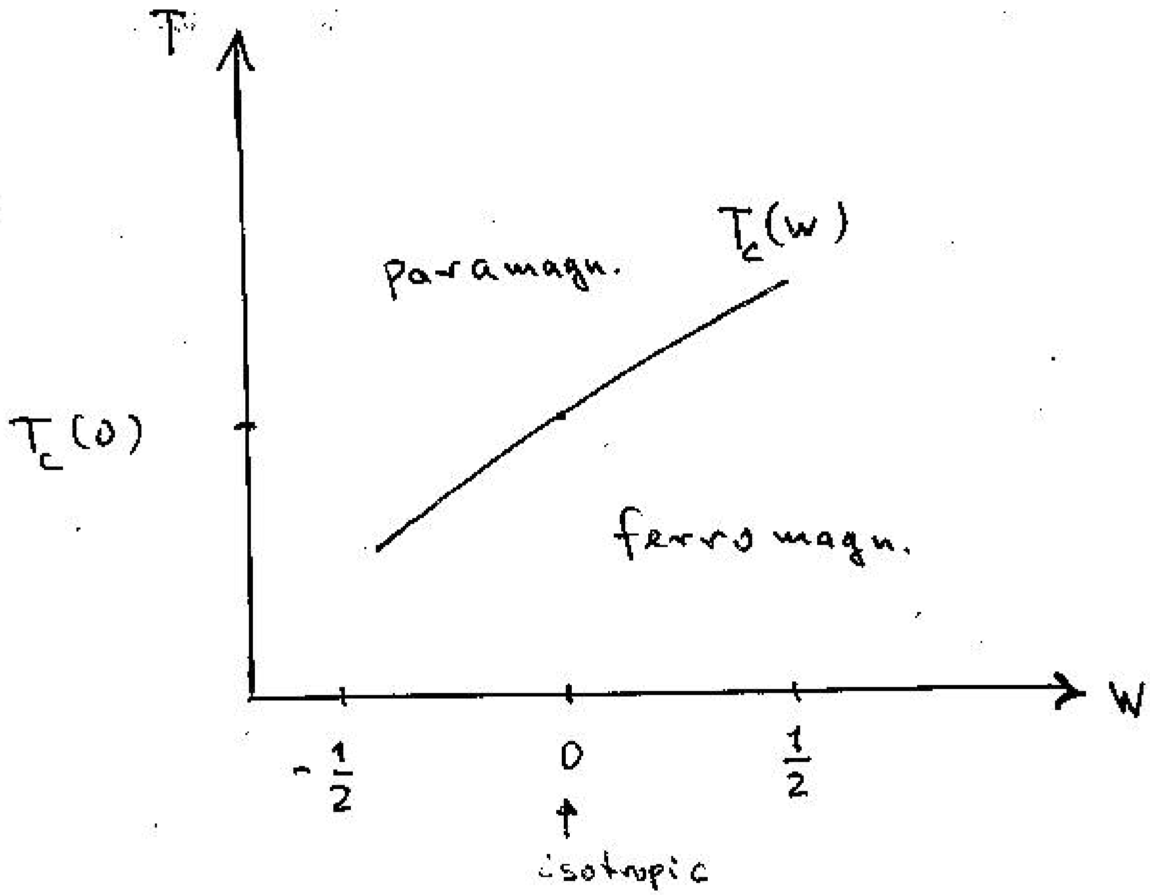
next-nearest-neighbor coupling $\quad J' \quad$ anisotropic
in each plane

anisotropy matrix

$$(A_{\alpha\beta}) = c_0 \begin{pmatrix} 1 & w & w \\ w & 1 & w \\ w & w & 1 \end{pmatrix}$$

$$-\frac{1}{2} < w = \frac{J'}{J + 2J'} \leq \frac{1}{2}$$

Phase diagram of anisotropic model



Ferromagnet. crit. point : $T_c(0)$ für $J > 0$, $J' = 0$

" line of crit. points : $T_c(w)$ für $J > 0$, $J' \neq 0$

3 eigenvalues of $(A_{\alpha\beta})$ positive for $-\frac{1}{2} < w \leq \frac{1}{2}$

$\boxed{\text{Critical exponents : independent of } w}$

<u>But :</u> correlation function and $\boxed{\text{finite-size-effects}}$

$\boxed{\textbf{w dependent}}$

Amplitude of free energy

$$f_{sing}(t,h,L) = L^{-3}\, Y(C_1 t L^{1/\nu}\,,\, C_2 h L^{\beta\delta/\nu})$$

universal finite-size scaling function Y

at criticality : $t = 0\,,\ h = 0$:

Privman - Fisher hypothesis : universal

$$f_{sing}(0,0,L) = L^{-3}\, Y(0,0)$$

Check : anisotropic φ^4-theory

exact result for $n \to \infty$ [X.S. Chen, V. Dohm, Phys. Rev. E **70**, 056136 (2004)

$\boxed{\text{$Y(0,0)$ depends on anisotropy matrix}}$

Ising Model in finite volume

$$H = - \sum_{i,j} J_{ij}\, \sigma_i\, \sigma_j - h \sum_i \sigma_i$$

$$M = \frac{1}{N} \sum \sigma_i$$

dimensionless ratio ("Binder cumulant")

$$U(t,L) = -\frac{1}{3} \frac{< M^4 > - 3 < M^2 >^2}{< M^2 >^2}$$

large L : $= U(ctL^{1/\nu})$ scaling form, c nonuniversal

$T = T_c$: $\lim_{L \to \infty} U(0,L) = U^* =$ **universal**

independent of J_{ij} , lattice structure ,

equivalent : via free energy

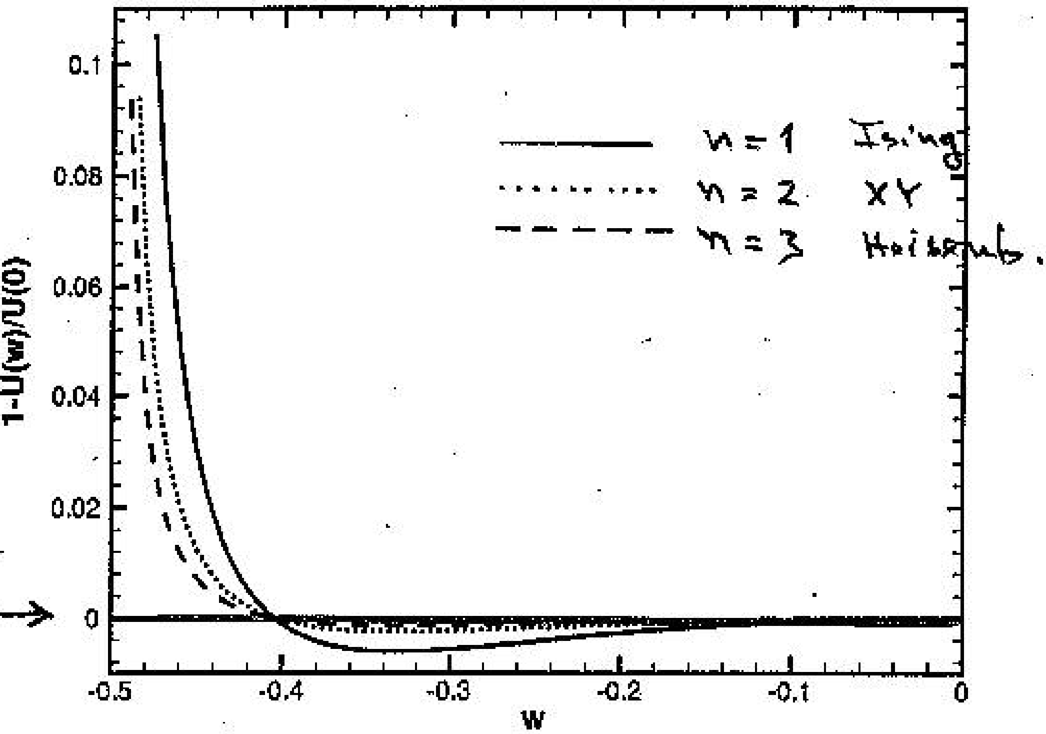
$$f_{sing}(t,h,L) = L^{-d}\, Y\, (C_1 t L^{1/\nu}\, ,\, C_2 h L^{\beta\delta/\nu})$$

$$U^* = \frac{1}{3} \frac{\frac{\partial^4}{\partial y^4} Y(0,y)}{\left[\frac{\partial^2}{\partial y^2} Y(0,y)\right]^2} \Bigg|_{y=0}$$

Quantitative prediction

Relative anisotropy effect on Binder cumulant at T_c

$d = 3$



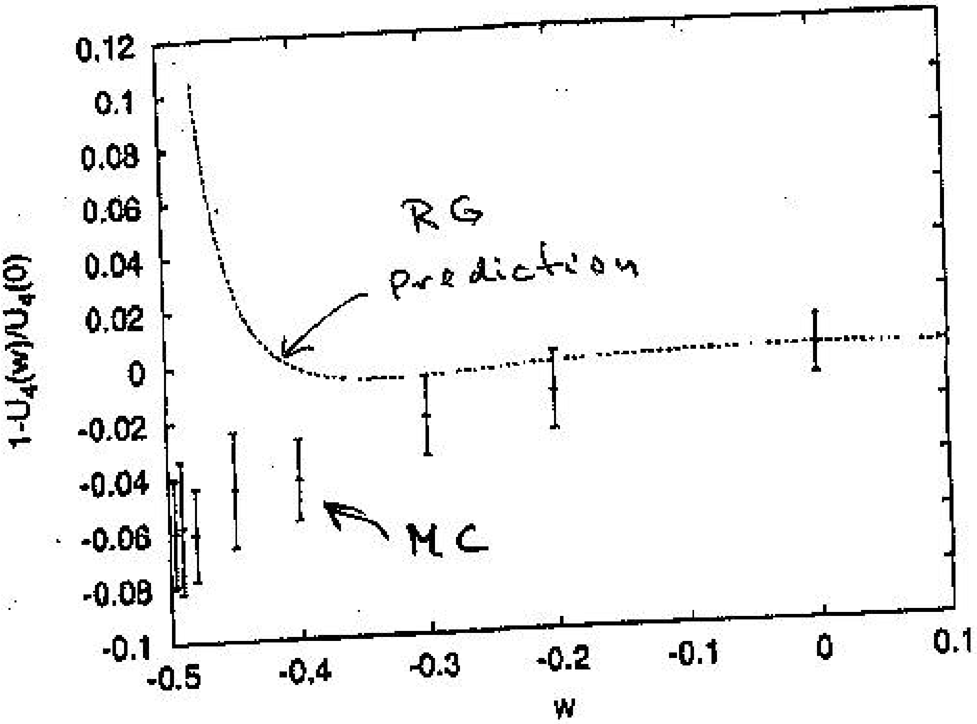
isotropy →

X.S. Chen, V. Dohm, Phys. Rev. E $\underline{70}$, 056136 (2004)

Challenge to MC simulations !

small : 1% effect

w



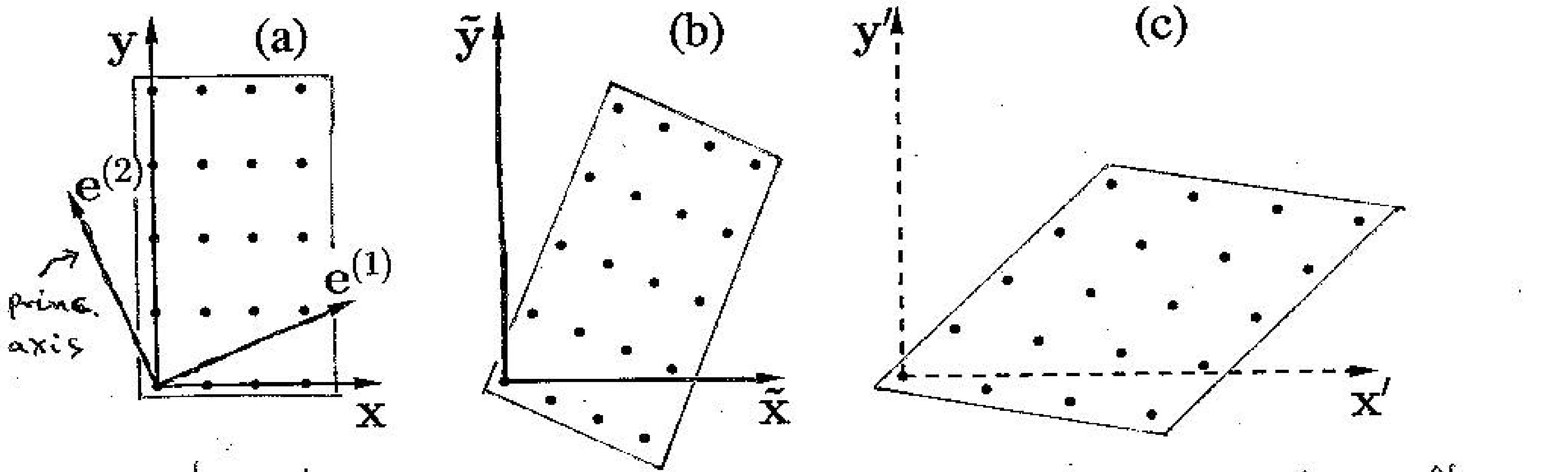
$d = 3$ Ising model

MC simulations

M. Schulte, C. Drope, Int. J. Mod. Phys. C **16**, 1217 (2005)

Exact relation between anisotropic and isotropic systems

anisotropic system → $\boxed{\text{rotation}}$ anisotropic system → $\boxed{\text{rescaling}}$ in the direction of the principle axes <u>isotropic</u> system

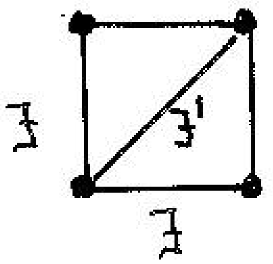


rectangular shape — rectangular shape — <u>non-rectangular</u> shape

$$\underline{\underline{A}} \qquad \underline{\underline{A}}' = U \underline{\underline{A}} U^{-1} = \underline{\underline{\lambda}} \qquad \underline{\underline{\tilde{A}}} = \underline{\underline{\lambda}}^{-1/2} \underline{\underline{A}}' \underline{\underline{\lambda}}^{-1/2} = \mathbb{1}$$

non-diagonal — diagonal $\lambda_1 \neq \lambda_2 \neq \lambda_3$ — isotropic

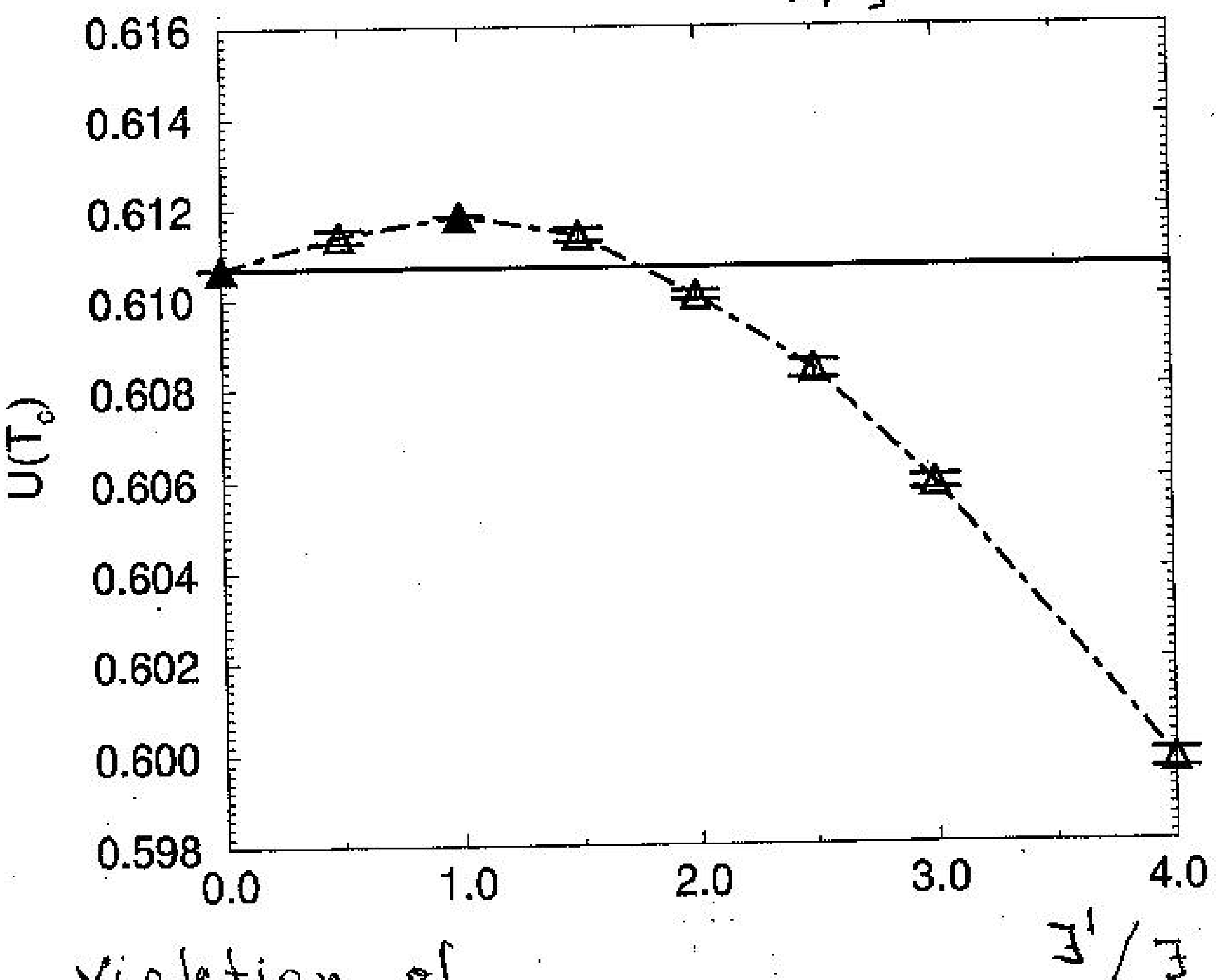
$d = 2$ square Ising model



MC simulations:

W. Selke, L. N. Shchur, J. Phys. A 38, L 739 (2005)

Binder cumulant U : nonuniversal dependence on J'/J



Violation of

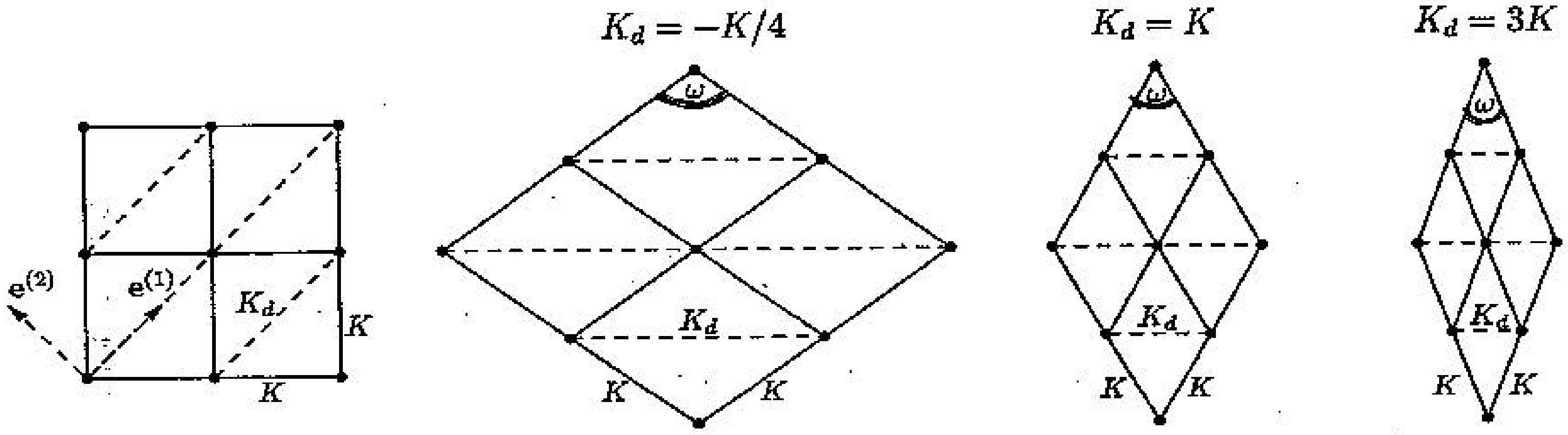
Prediction for anisotropic square Ising models

[V. Dohm, J. Phys. A 39, L 259 (2006)]

Variation of diagonal coupling K_d of anisotropic system is equivalent to variation of shape of isotropic system

$$\boxed{\omega = 2 \arctan\left\{ \left[1 + \frac{2K_d}{K} \right]^{-1/2} \right\}}$$

family of Ising models in non-rectangular geometry

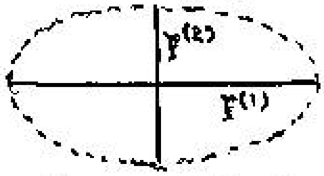


(a) (b) (c) (d)

Same Binder cumulant as anisotr. models in rect. geom correlation volume:

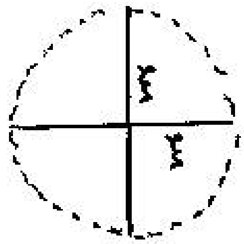
$$V_{corr} = \prod_{i=1}^{d} \xi^{(i)} \quad , \quad \text{anisotropic} \neq \xi^{d} .$$

anisotropic



ellipsoidal

isotropic



spherical

$$f_s(t; \mathbf{A}, u_0) = (\det\ \mathbf{A})^{-1/2}\, f_s'(t; u_0')$$

bulk property

$$\boxed{\lim_{t \to 0+} f_s(t; \mathbf{A}, u_0) \prod_{i=1}^{d} \xi^{(i)} = Q(d,n) = \text{universal}}$$

same constant as for isotropic systems within the same univ. class

instead of

$$f_s \cdot \xi^d = Q$$

Summary

(1) Critical exponent α and amplitude ratio A^+ / A^- of ^{4}Helium

in approximate agreement with RG and MC,

but small unresolved discrepancies

(2) "2-scale factor universality" is

not valid

for anisotropic systems of noncubic symmetry
neither for <u>bulk</u> nor for <u>confined</u> systems.

(3) Finite-size scaling functions $Y(x, y)$ are

nonuniversal

for anisotropic systems of noncubic symmetry
in a given rectangular geometry — confirmed by MC for $d = 2$

(4) **Nonuniversal** finite-size scaling functions in **anisotropic** systems with **rectangular** shape

are exactly related to

universal finite-size scaling function of **isotropic** systems with **non-rectangular** shape

(5) quantitative predictions for anisotropic spin models :

$$Y(0,0) \; ; \quad U^* \; ; \quad \omega$$

[illegible] MC simulations