Not everything is Gaussian !!!

M.A. Nowak, J. Jurkiewicz (Jagellonian University)

G. Papp (Eötvös Loránd University)

I. Zahed (Stoony Brook, SUNY)

Not everything is Gaussian !!!

M.A. Nowak, J. Jurkiewicz (Jagellonian University)

G. Papp (Eötvös Loránd University)

I. Zahed (Stoony Brook, SUNY)

Outline:

- Gaussian random matrices
- Stable distributions in classical probability
- Lévy's domain of attraction
  - free Lévy random variables
  - Wigner-Lévy ensemble
- Summary

Gaussian random matrices

$N \times N$ real symmetric matrices $H = H^T$, $N \to \infty$

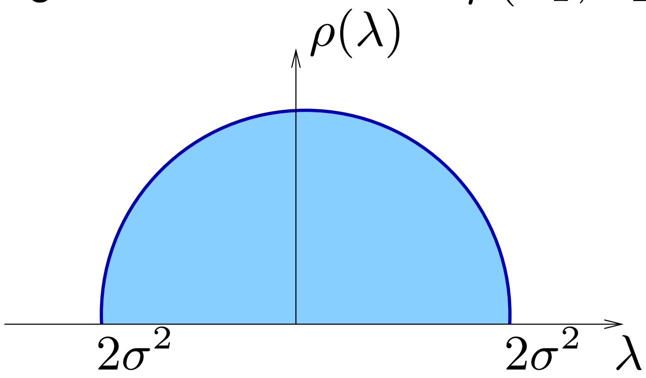
$$H = \begin{pmatrix} h_{11} & h_{12} & h_{13} & \dots & h_{1N} \\ & h_{22} & h_{23} & \dots & h_{2N} \\ & & h_{33} & \dots & h_{3N} \\ & & & \dots & \dots \\ & & & & h_{NN} \end{pmatrix}$$

Wigner: h_{ij} i.i.d. $\mathcal{N}\left(0, \frac{\sigma^2}{N}\right)$

GOE: $Z = \int dH \exp -\frac{N}{2\sigma^2} \mathrm{Tr} H^2$

Questions:

eigenvalue density $\rho(\lambda)$, eigenvalue correlations $\rho(\lambda_1, \lambda_2)$, largest eigenvalue etc.



Applications

- many-body quantum systems
- localization theory
- complex systems (random free-energy landscapes)
- quantum chaos
- QCD - Dirac operator
- color expansion $1/N$ (planar diagrammatics)
- counting (knots, pseudoknots enumeration)
- 2d quantum gravity, non-critical strings
- mulitvariate analysis
- telecommunication
- quantitative finance

Non-Gaussian Wigner ensemble

$N \times N$ real symmetric matrices $H = H^T$, $N \to \infty$

$$H = \begin{pmatrix} h_{11} & h_{12} & h_{13} & \dots & h_{1N} \\ & h_{22} & h_{23} & \dots & h_{2N} \\ & & h_{33} & \dots & h_{3N} \\ & & & \dots & \dots \\ & & & & h_{NN} \end{pmatrix}$$

h_{ij} i.i.d. , $E(h_{ij}) = 0$, var$(h_{ij}) = \sigma^2 < \infty \implies$ Gaussian universality

h_{ij} i.i.d. , $E(h_{ij}) = 0$, var$(h_{ij}) = \infty \implies$ Lévy universality

Similarly as in CLT

Laws of addition in classical probability

Let x_1, x_2 be independent random variables with pdfs: $p_1(x_1), p_2(x_2)$

What is pdf for: $x = x_1 + x_2$?

$p_{1+2}(x) = \int \mathrm{d}x_1 p_1(x_1) p_2(x - x_1)$

Laws of addition in classical probability

Let x_1, x_2 be independent random variables with pdfs: $p_1(x_1), p_2(x_2)$

What is pdf for: $x = x_1 + x_2$?

$$p_{1+2}(x) = \int \mathrm{d}x_1 p_1(x_1) p_2(x - x_1)$$

Characteristic function:

$$\phi(k) = \int \mathrm{d}x \; p(x) e^{ikx}$$

$$\phi_{1+2}(k) = \phi_1(k) \cdot \phi_2(k)$$

Laws of addition in classical probability

Let x_1, x_2 be independent random variables with pdfs: $p_1(x_1), p_2(x_2)$

What is pdf for: $x = x_1 + x_2$?

$p_{1+2}(x) = \int \mathrm{d}x_1 p_1(x_1) p_2(x - x_1)$

Characteristic function:

$\phi(k) = \int \mathrm{d}x \; p(x) e^{ikx}$

$\phi_{1+2}(k) = \phi_1(k) \cdot \phi_2(k)$

Cumulants' generating function:

$c(k) = \log(\phi(k))$

$c_{1+2}(k) = c_1(k) + c_2(k)$

Stable distributions:

$X = x_1 + \ldots + x_N$ sum of iids $\Longrightarrow c_N(k) = Nc(k)$

Important class:

$c(k) = -a|k|^\mu \Longrightarrow c_N(k) = Nc(k) = c(N^{1/\mu}k)$

$\phi(k) = e^{-a|k|^\mu}$

Stable laws:

$p(x) = \int \frac{dk}{2\pi} e^{-ikx} e^{-\gamma^\mu |k|^\mu}$, for $\mu \in (0, 2]$

In this case:

$X = \frac{x_1 + \ldots + x_N}{N^{1/\mu}}$ is identically distributed as x_i

Stable laws are like fixed points in RG !

$L_\mu(x) \sim |x|^{-1-\mu}$, e.g. $L_1(x) = \frac{1}{\pi} \frac{1}{1+x^2}$

Lévy ensembles

Class 1: LOE

Class 2: Lévy-Wigner ensemble

h_{ij} iid , pdf $p(h_{ij}) \sim \frac{1}{h_{ij}^{\mu+1}} \quad 0 < \mu < 2$

$\mathcal{H} = \frac{H_1 + \cdots + H_n}{n^{1/\mu}}$

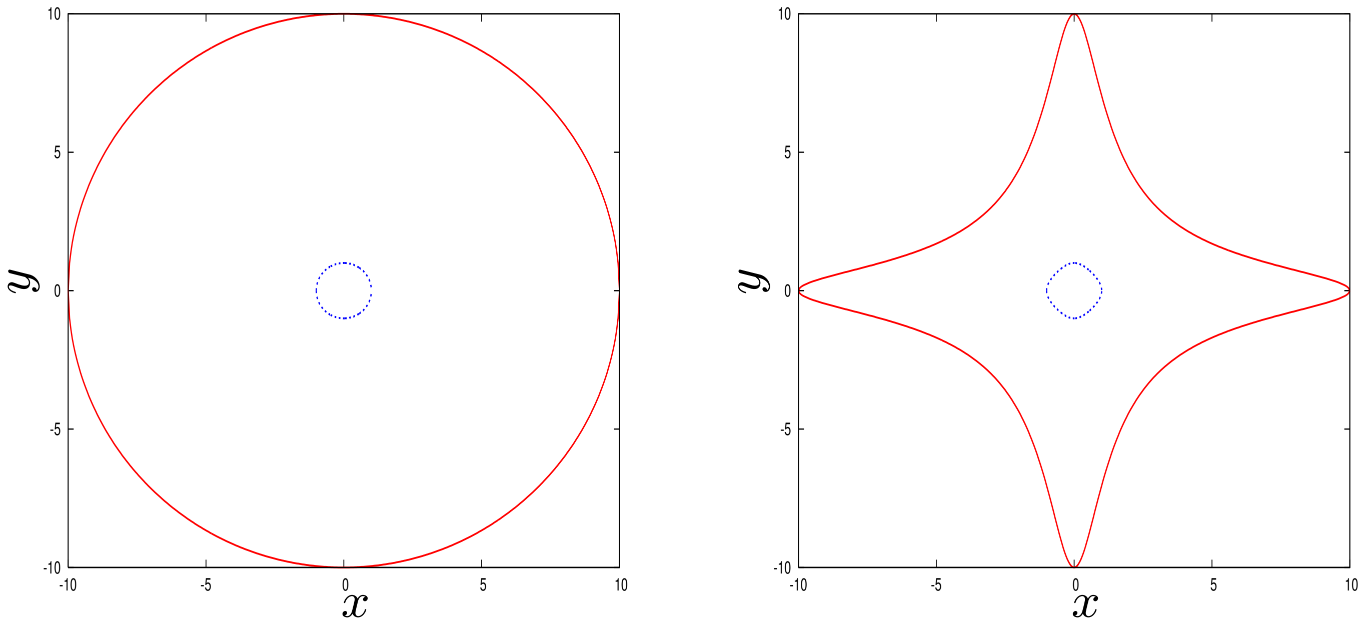
Class 1: $\mathcal{H}$ free random matrices with stable sdf

Class 2: $\mathcal{H}$ pdf $p(h_{ij}) \to L_\mu(h_{ij})$

Simple example for 2d-random vectors:

pdfs for (x, y)

$$p_1(x,y) \sim e^{-x^2} e^{-y^2} \qquad p_2(x,y) \sim \frac{1}{1+x^2}\,\frac{1}{1+y^2}$$



$$p(x,y) = \text{const}$$

Class 1 = free random variables

Free probability (Voiculescu)

Assymptotic freeness of large matrices from class 1 (Speicher)

Classical probability

$p(x)$ - pdf

$\phi(k) = \int \mathrm{d}x p(x) e^{+ikx}$

$c(k) = \log \phi(k)$

$c_{1+2}(k) = c_1(k) + c_2(k)$

$\phi(k) = \exp c(k)$

$p(x) = \frac{1}{2\pi} \int \mathrm{d}k \phi(k) e^{-ikx}$

Free probability

$\rho(\lambda)$ - sdf

$G(z) = \int \frac{\rho(\lambda)\mathrm{d}\lambda}{\lambda - z}$

$G(R(z) + \frac{1}{z}) = z$

$R_{1+2}(z) = R_1(z) + R_2(z)$

$R(G(z)) + \frac{1}{G(z)} = z$

$\rho(\lambda) = -\frac{1}{\pi} \mathrm{Im}\, G(\lambda + i0^+)$

Free lunch

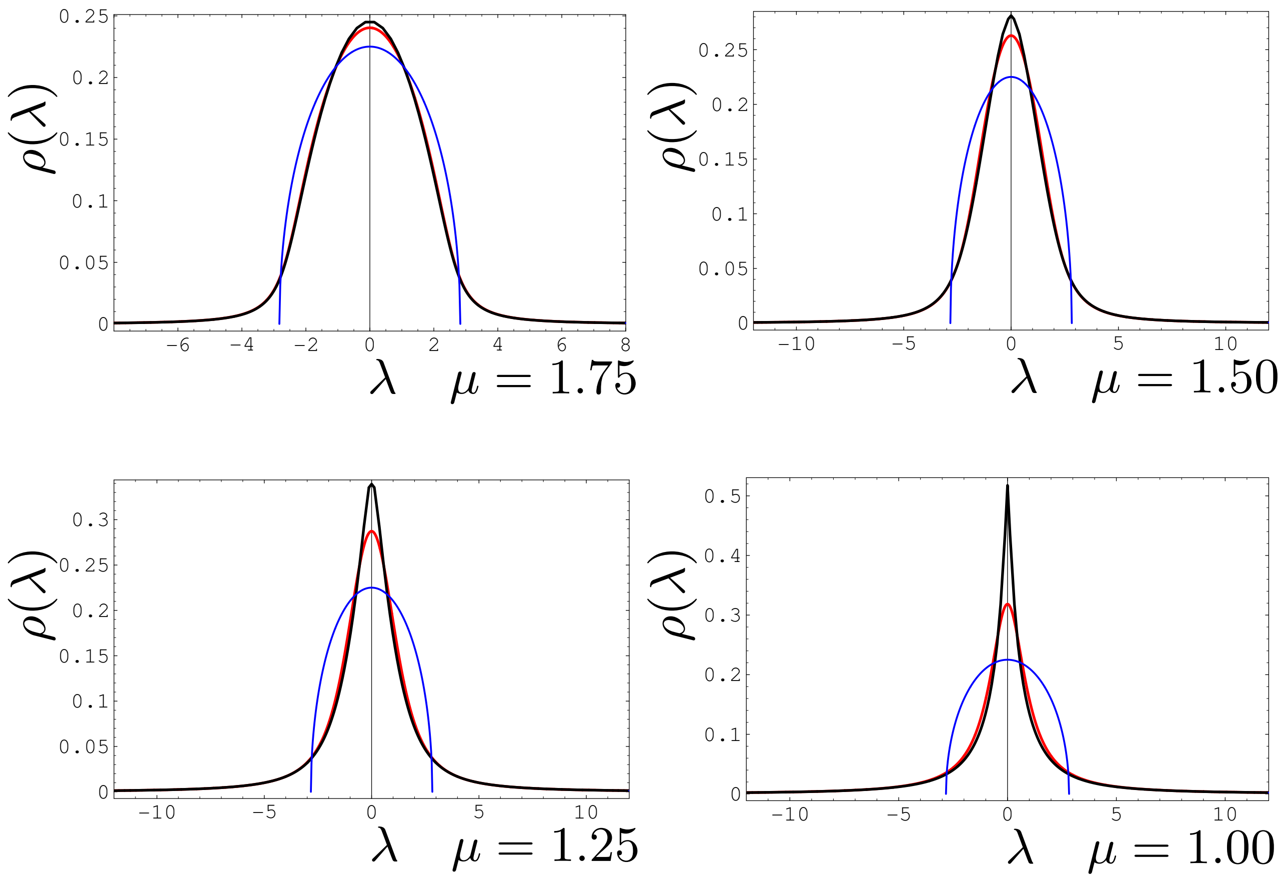
Bercovici-Pata bijection between classical and free random variable

Stable laws in free probability (Bercovici-Voiculescu)

Class 2 - Wigner-Lévy

Bouchaud-Cizeau, (tour de force !!)

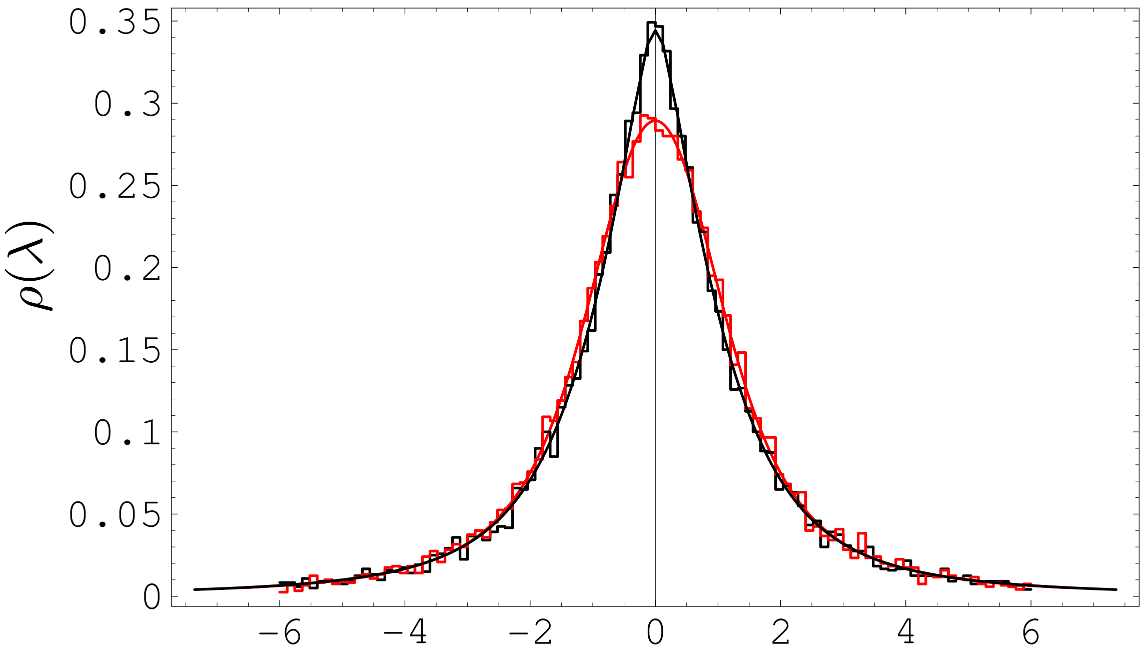
ZB, M.A.Nowak, J.Jurkiewicz, G. Papp, I. Zahed



From Wigner-Lévy to free random Lévy

H_i random Wigner-Lévy $\quad O_i$ random orthogonal matrices

$$\mathcal{H} = \frac{1}{n^{1/\mu}} \sum_{i=1}^{n} O_i H_i O_i^{\tau}$$



$n = 100, \mu = 1.25$

Summary

- Wigner-Lévy and free Lévy (LEO) ensembles
- spectral denisty can be calculated in both cases
- CLT for randomly rotated Wigner-Lévy give free Lévy