Problem Set XII

Advanced Statistical Physics – SoSe 2017
Due: Tuesday, July 4, before the lecture PROF. DR. KLAUS KROY

Exercise 1. Elimination of Fast Variables (+15 P.)
Consider the one dimensional Hamiltonian:

$$H = \frac{P^2}{2M} + V(X) + \sum_{\lambda=1}^{N} \frac{p_\lambda^2}{2m_\lambda} + \frac{1}{2}\sum_{\lambda=1}^{N} m_\lambda \omega_\lambda^2 \left(x_\lambda - \frac{\gamma_\lambda}{m_\lambda \omega_\lambda^2} X \right)^2 \tag{1}$$

for a particle of mass M linearly coupled with N harmonic oscillator of masses m_λ such that $M \gg m_\lambda \,\forall\lambda$. X and P are the position and momentum of the heavy particle, m_λ, x_λ, p_λ, and ω_λ the masses, positions, momenta and frequencies of the light particles respectively. γ_λ gamma controls the coupling strength between the λ-th light particle and the heavy one. The elimination of the variables x_λ and p_λ from the equation of motion associated to (1) leads to a Brownian motion for the heavy particle of the form:

$$\dot{P}(t) = -V'\left(X(t)\right) - \int_0^t \zeta\left(t-t'\right) \dot{X}\left(t'\right) dt' + \xi(t) \tag{2}$$

1.1 Find the expression for $\zeta(t-t')$ and $\xi(t)$. Check if time-reversal symmetry satisfied $(t \to -t)$ is satisfied? (**+6 P.**)

1.2 Study the first two moments of the noise (i.e. $\langle \xi(t) \rangle$ and $\langle \xi(t)\,\xi(t') \rangle$).
Hint: Assume that the initial values $x_\lambda(0)$ and $p_\lambda(0)$ are distributed according to a Maxwell-Boltzmann distribution:

$$P\left(x_\lambda(0), p_\lambda(0)\right) \propto \exp\left\{ -\beta \sum_{\lambda=1}^{N} \left[\frac{p_\lambda^2(0)}{2m_\lambda} + \frac{m_\lambda \omega_\lambda^2}{2} \left(x_\lambda(0) - \frac{\gamma_\lambda}{m_\lambda \omega_\lambda^2} X(0) \right)^2 \right] \right\} \tag{3}$$

Argue why such a distribution is a good choice. (**+3 P.**)

1.3 Is the fluctuation-dissipation relation $\langle \xi(t)\,\xi(t') \rangle = k_B T \zeta(t-t')$ satisfied? (**+2 P.**)

1.4 The Langevin equation (2) is called generalized (or with memory) since the evolution depends on all the previous times via the integral. Discuss the Markov limit, i.e. $\zeta(t-t') \simeq 2\gamma\delta(t-t')$ of the model in terms of the parameters omega and gamma. Is time-reversal symmetry still satisfied? If it is not, find a physical explanation. (**+4 P.**)