Problem Set X

Advanced Statistical Physics – SoSe 2017

Due: Tuesday, June 13, before the lecture PROF. DR. KLAUS KROY

Exercise 1. Renormalization group transformation 12 P.

Consider the Ising model in 2 dimensions on a square lattice. The partition sum for N Ising spins s_i $(i = 1, \ldots, N)$ is given by

$$Z_N(K_0) = \sum_{\{s_i\}} \exp\left(K_0 \sum_{\langle i,j \rangle} s_i s_j\right),$$

where $K_0 = \beta J$ is the coupling constant and $\langle i, j \rangle$ symbolizes the sum over nearest neighbors.

1.1 Next-nearest neighbors form two diagonal sublattices. In a first renormalization step perform a spin decimation. Consider a spin s in one sublattice and its nearest neighbors s_1, s_2, s_3, s_4 on the other sublattice. Show that

$$\sum_{s=\pm 1} e^{K_0 s(s_1+s_2+s_3+s_4)} = A \exp\left(K' \sum_{1 \le i < j \le 4} s_i s_j + U s_1 s_2 s_3 s_4\right)$$

holds and determine A, K' and U.
Taking the partition sum over all spins $s = \pm 1$ of the first sublattice halves the number of degrees of freedom, and a renormalized Hamiltonian can be defined, which contains additional interactions. The original vector of the coupling constants $\mathbf{K}_0 = (K_0, 0, 0, 0, \ldots)$ is transformed to $\mathbf{K}_1 = (K_1, L_1, U_1, 0, \ldots)$ with the new nearest neighbor interaction K_1, next-nearest neighbor interaction L_1, and four-spin coupling U_1. Establish the relation between K_1, L_1, U_1 and K', U, K_0. (**8 P.**)

1.2 Since a systematic implementation of further RGT steps is very complex, assume in the following that $K_0 \ll 1$ and consider only the leading order of K_n. Determine the RG–equations $K_1(K_0)$ and $L_1(K_0)$, and the four-spin coupling constant U_1 for the first renormalization step. (**4 P.**)

Exercise 2. Scaling, metrices and dimensions (+4 P.)

Harpy eagles and squirrel monkeys are natural enemies and often compete in races.

2.1 Consider such a race in 2 dimension in the urban jungle of Manhattan. They are both starting at the same time and move with the same velocity from the start to the end point as directly as possible. However, the squirrel monkey walks and is thus forced to use a Manhattan metric:

$$||\mathbf{r}||_M = \sum_i |r_i| \tag{1}$$

, while the harpy eagle flies using the Euclidean metric:

$$||\mathbf{r}||_E = \sqrt{\sum_i r_i^2}. \tag{2}$$

Who will be the first at their destination? Justify your decision with a calculation. (**+1 P.**)

2.2 Now consider the same race in a down-scaled model of Manhattan which is 1/10 of the size of the original. Does this change produce any benefit for one of the two animals?(**+1 P.**)

2.3 Do your decisions change, when the race is held in 3 dimensions? (**+2 P.**)