Problem Set IX

Advanced Statistical Physics – SoSe 2017

Due: Tuesday, June 6, before the lecture

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Exercise 1. 1d-Ising model 5 P.

The Hamiltonian of the one-dimensional Ising model for N spins ($s_i = \pm 1$, $i = 1, \ldots, N$) with open boundary conditions is given by

$$H = -\sum_{i=1}^{N-1} J_i s_i s_{i+1} \,.$$

J_i is the interaction energy of the neighboring spins at i and $i+1$.

1.1 Calculate the partition sum

$$Z_N = \sum_{\{\mathbf{s}\}} e^{-H(\{\mathbf{s}\})/(k_B T)},$$

by deriving a recurrence relation between Z_N and Z_{N+1}. Determine Z_N for the common special case $J_i \equiv J \;\forall i$.

1.2 Calculate the spin correlation function $\langle s_i s_{i+j} \rangle$ and specialize the result again for $J_i \equiv J \;\forall i$.

1.3 Show that the spontaneous magnetization $M_s = \langle s \rangle$ (for homogeneous interactions $J_i \equiv J$) can only take two values for the infinitely large system:

$$M_s(T) = \begin{cases} 0\,, & T > 0 \\ 1\,, & T = 0 \end{cases} .$$

Hint: Use the ansatz $h(r) = r^{-\alpha} \mathrm{e}^{-r/\xi}$. Make use of $\langle s_i s_{i+j} \rangle \overset{j\to\infty}{\longrightarrow} \langle s \rangle^2$.

Exercise 2. Magnet and scaling hypothesis 10 P.

Consider a ferromagnet with magnetizaion work per volume $\delta w = \delta W/V = \mu_0\, H\, \mathrm{d}M$, and assume the internal energy per volume $u = u(T, M)$ and the magnetization $M = M(T, H)$ to be given.

2.1 Show that the specific heats at constant M and H obey the thermodynamic relation

$$c_M - c_H = \mu_0\, T \left(\frac{\partial H}{\partial T}\right)_M \left(\frac{\partial M}{\partial T}\right)_H . \tag{1}$$

Hint: Use appropriate Maxwell relations.

2.2 Calculate $c_M - c_H$ using the equation of state (for small H, $M \geq 0$)

$$H = \frac{1}{C}(T - T_c)M + bM^3, \tag{2}$$

where C, T_c and b are positive constants.

2.3 Use the result from 2.2 to show that $c_M/c_H < 1$ holds for $T \to T_c^{(-)}$.

2.4 Show that the equation 1 can be written as

$$c_H - c_M = \mu_0\, T \left(\frac{\partial M}{\partial T}\right)_H^2 \left(\frac{\partial M}{\partial H}\right)_T^{-1},$$

and use this to derive the exponent exponent relation $2\beta + \gamma + \alpha = 2$ assuming that the inequality shown in 2.3 is valid.

2.5 Assume that the singular part of the generalized free energy $\mathcal{F}_s(t, h)$ of a magnet is 2 times continuously differentiable and satisfies the generalized homogeneity relation

$$\mathcal{F}_s(t, h) = \lambda^{-1}\mathcal{F}_s(\lambda^{a_t} t, \lambda^{a_h} h),$$

where λ is a positive scale factor, $t = (T - T_c)/T_c$ and h an external magnetic field. Express the critical exponents α, β and γ in terms of a_t and a_h and derive the exponent relation obtained in 2.4.