Problem Set VIII

Advanced Statistical Physics – SoSe 2017
Due: Tuesday, May 30, before the lecture PROF. DR. KLAUS KROY

Exercise 1. Ginzburg-Landau theory: Correlation function 12 P.
The Ginzburg-Landau functional is defined by

$$L \equiv n_c k_B T_c \int_V d\mathbf{r}\, \mathcal{L}_G \,, \qquad \text{where} \qquad \mathcal{L}_G = \frac{\ell^2}{2}(\nabla\psi)^2 + \frac{t}{2}\psi^2 + \frac{g}{4}\psi^4 \,.$$

1.1 Show that, the Fourier transform $h_{\mathbf{q}} \equiv \langle|\psi_{\mathbf{q}}|^2\rangle$ of the pair correlation function $h(\mathbf{r})$ in harmonic approximation, obeys

$$(|\mathbf{q}|^2 + \xi^{-2})h_{\mathbf{q}} = C,$$

where $C = T/(T_c n_c l^2)$ and $\xi \equiv l|t|^{-1/2}$ denotes the correlation length of the fluctuations in the order parameter. (**6 P.**) Proceed as follows:

i) Show that

$$L = \frac{n_c k_B T_c}{2} \sum_{\mathbf{q}} \left(\ell^2 q^2 + t\right) |\psi_{\mathbf{q}}|^2.$$

Hint: Use the definitions $\psi_{\mathbf{q}} = \frac{1}{\sqrt{V}} \int_V d\mathbf{r}\, \psi(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}}$, $\psi(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{q}} \psi_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}$ and show the identity $\delta_{\mathbf{q},\mathbf{q}'} = \frac{1}{V}\int_V d\mathbf{r}\, e^{-i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{r}}$.

ii) Show that the complex Fourier components $\psi_{\mathbf{q}}$ are not suitable as independent degrees of freedom in the partition sum. (Hint: Decompose into real and imaginary parts.) Resolve the problem by restricting the modes to an appropriate $\mathbf{q}$-range.

iii) Calculate the average value

$$\langle|\psi_{\mathbf{q}}|^2\rangle = \frac{\int_{\mathbb{C}} d\psi_{\mathbf{q}}\, |\psi_{\mathbf{q}}|^2 \exp\left\{-\beta n_c k_B T_c \left(\ell^2 q^2 + t\right) |\psi_{\mathbf{q}}|^2\right\}}{\int_{\mathbb{C}} d\psi_{\mathbf{q}} \exp\left\{-\beta n_c k_B T_c \left(\ell^2 q^2 + t\right) |\psi_{\mathbf{q}}|^2\right\}},$$

by decomposing $\psi_{\mathbf{q}}$ into real and imaginary parts.

1.2 Use the result from 1.1 to show that $h(\mathbf{r})$ obeys the partial differential equation

$$\nabla^2 h(\mathbf{r}) - \xi^{-2} h(\mathbf{r}) = -C\delta(\mathbf{r}). \quad \textbf{(2 P.)} \qquad (1)$$

1.3 For $h(\mathbf{r}) = h(r)$, $\mathbf{r} \in \mathbb{R}^d$ and $r \equiv |\mathbf{r}| > 0$ write the equation 1 in the following form

$$\frac{\mathrm{d}^2 h}{\mathrm{d}r^2} + \frac{d-1}{r}\frac{\mathrm{d}h}{\mathrm{d}r} - \frac{h}{\xi^2} = 0. \quad (\mathbf{2\ P.}) \tag{2}$$

1.4 Calculate $h(\mathbf{r})$ in the limits $\xi \to \infty$ and $r \to \infty$ (**2 P.**)
Hint: Use the ansatz $h(r) = r^{-\alpha}\mathrm{e}^{-r/\xi}$ for solving equation 2.