Problem Set VII

Advanced Statistical Physics – SoSe 2017
Due: Tuesday, May 23, before the lecture

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Exercise 1. Poisson-Boltzmann equation 10 P.

Consider a globally neutral symmetric plasma consisting of particles with charges $z_\alpha e$ ($\alpha = \pm$) with valences $z_+ = -z_- = z$ and densities $\int \mathrm{d}\mathbf{r}\, n_\pm(\mathbf{r})/V \equiv n_\pm = n/2$.

1.1 For the electrostatic potential

$$\phi_\alpha(\mathbf{r}) = \sum_{\beta=\pm} \int d^3r' \frac{z_\alpha z_\beta l_\mathrm{B}}{|\mathbf{r}-\mathbf{r}'|} n_\beta(\mathbf{r}')$$

the relation $\phi_+(\mathbf{r}) = -\phi_-(\mathbf{r}) \equiv \phi(\mathbf{r})$ holds, where $l_\mathrm{B} \equiv e^2/k_\mathrm{B}T$ is the so-called *Bjerrum length*. Derive the Poisson-Boltzmann equation $\Delta\phi(\mathbf{r}) = 4\pi z^2 l_\mathrm{B} n \sinh\phi(\mathbf{r})$ from the extremum principle for the RPA density functional by showing that $n_\pm(\mathbf{r}) = n\exp[\mp\phi(\mathbf{r})]/2$ holds for the charge carrier density distribution. (**5 P.**)

1.2 Solve the Poisson-Boltzmann equation for a neutral solution of ions in the half space $z > 0$ above an infinitely extended charged surface. Calculate and sketch $n_\pm(z)$. (**5 P.**)
Hint: Rewrite the Poisson-Boltzmann equation so that it is of the from:

$$\frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 - \cosh\phi(t) \equiv E,$$

where E is a constant that can be determined from a boundary condition.

Exercise 2. Benford's law (+4 P.)

To get on to the track of tax evaders, revenue authorities occasionally exploit the fact that the significant numbers of many empirical statistical series are *not* equally distributed over $\{1,\ldots,9\}$, but rather follow another universal law. Derive the form of this law by assuming that it remains invariant under a change of units.
Hint: Express the numbers in the form $m \times 10^n$ with $m \in [1/10, 1)$ and consider the probability distribution of m.