Problem Set III

Advanced Statistical Physics – SoSe 2017
Due: Tuesday, April 25, before the lecture Prof. Dr. Klaus Kroy

Exercise 1. Structure of dilute gases 10 (+4) P.

Consider a classical gas of particles interacting via the "statistical" pair potential

$$\nu_{\text{eff}}(\mathbf{r}_i - \mathbf{r}_j) = -k_{\text{B}}T \ln[1 - \delta \exp(-2\pi(\mathbf{r}_i - \mathbf{r}_j)^2/\lambda_T^2)]$$

which constitues a simple model for a dilute neutral quantum gas ($\delta = +1$ for fermions and $\delta = -1$ for bosons).

1.1 Calculate and sketch the corresponding radial distribution function $g(\mathbf{r}) = V\langle\delta(\mathbf{r} - \mathbf{r}_1' + \mathbf{r}_2')\rangle$ in the low-density limit. (**2 P.**)

1.2 Calculate and sketch the corresponding structure factor $S_{\mathbf{q}}$ which is defined as the three-dimensional Fourier transform of $g(\mathbf{r}) - 1$. (**6 P.**)

1.3 The second virial coefficient

$$B(T) \equiv -\frac{1}{2} \int_{\mathbb{R}^3} \mathrm{d}\mathbf{r} \left[\mathrm{e}^{-\beta\nu(\mathbf{r})} - 1\right]$$

provides the leading corrections to the ideal gas law due to a pair interaction potential $\nu(\mathbf{r})$ between the particles. Calculate $B(T)$ for the statistical potential ν_{eff} (**1 P.**) as well as for a dilute gas of hard spheres (**1 P.**) of diameter σ for which the pair potential is given by

$$\nu(\mathbf{r}_i - \mathbf{r}_j) = \begin{cases} \infty, & |\mathbf{r}_i - \mathbf{r}_j| < \sigma \\ 0, & |\mathbf{r}_i - \mathbf{r}_j| > \sigma \end{cases}.$$

1.4 Estimate the physical conditions under which the corrections due to the quantum nature of the gas become important by

a) comparing the two virial coefficients calculated in the previous task (**+2 P.**),

b) comparing the thermal wavelength λ_{T} with the typical distance over which the probability density $|\psi(r)|^2$ of a small atom (*e.g.* helium) decays to zero. (**+2 P.**)

Exercise 2. Number fluctuations and compressibility 4 P.

2.1 Using the grand canonical partition function $Z_{\mathrm{G}}(V,T,\mu) = \sum \exp(-\beta(E-\mu N))$, show that

$$\sigma_N^2 = k_{\mathrm{B}}T \left(\frac{\partial N}{\partial \mu}\right)_{T,V},$$

where $\sigma_N^2 \equiv \langle N^2\rangle - \langle N\rangle^2$. (**2 P.**)

2.2 Show that the isothermal compressibility $\kappa_T \equiv n^{-1}(\partial n/\partial p)_T$, where $n = N/V$, relates to σ_N^2 via

$$\kappa_T = \beta V \frac{\sigma_N^2}{N^2}.$$

Hint: Starting from the above definition of κ_T, use the Gibbs-Duhem relation $V\,\mathrm{d}p = S\,\mathrm{d}T + N\,\mathrm{d}\mu$ to express $\partial/\partial p$ in terms of $\partial/\partial\mu$. (**2 P.**)