The 3D-Abelian Higgsmodel –
Phase structure and photon propagator

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Introduction

vacuum expectation values $\leftarrow$ functional integrals

$$\langle 0|\mathcal{T}F[\Phi]|0\rangle_E = \frac{\int \mathcal{D}\phi F[\phi]\ e^{-S[\phi]}}{\int \mathcal{D}\phi\ e^{-S[\phi]}}$$

Close similarities to statistical physics:

euklidian action S	$\longrightarrow$	Hamiltonian (energy)
$\int \mathcal{D}\phi\ e^{-S[\phi]}$	$\longrightarrow$	partition function Z
$e^{-S[\phi]}$	$\longrightarrow$	Boltzmann weight

Our action: (Abelian Higgs model, scalar elektrodynamics, ...)

$$S^{SED} = \int d^d x \underbrace{\left[(D_\mu\phi)(D_\mu\phi^*) + m^2\phi\phi^* + g(\phi\phi^*)^2 + \frac{1}{4}F_{\mu\nu}F_{\mu\nu}\right]}_{\text{Lagrangian density}}$$

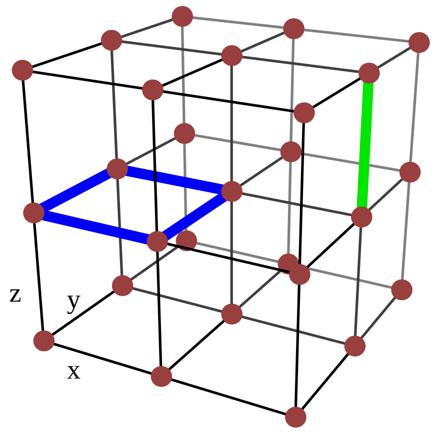
covariant derivative $D_\mu\phi = (\partial_\mu + ieA_\mu)\phi$	$\rightarrow$	gauge invariance
field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$	$\rightarrow$	curl of the em field

Outline

- Fields on a lattice
- Monte Carlo Methods
- Phase structure
- The photon propagator & the gauge fixing algorithm
- Summary

Discretization of the Higgs field

- d-dimensional (hyper-)cubic lattice
- linear extension L, lattice constant a
- periodic boundary conditions $\rightarrow$ (hyper-)torus



- $L \rightarrow \infty$
  thermodynamical limit
- $a \rightarrow 0$
  continuum limit

General idea: construct lattice action S_Γ s.t.

$$\boxed{S_\Gamma \xrightarrow[a\rightarrow 0]{} S}$$

use canonical lattice replacements:

$$\Delta_\mu \phi(x) = \frac{1}{a}(\phi(x + ae_\mu) - \phi(x))$$

$$\int d^d x \phi(x) \longrightarrow \int_\Gamma dx \phi(x) := a^d \sum_{x\in\Gamma} \phi(x)$$

Functional integral $\rightarrow$ high dimensional standard integral

$$\mathcal{D}\phi \rightarrow \prod_n d\phi(na)$$

Discretization of the Higgs field – cont.

In our case let $A_\mu = 0$ (Higgs field only)

$$S_H[\phi] = \int d^dx (\partial_\mu \phi)(\partial_\mu \phi^*) + m^2 \phi\phi^* + g(\phi\phi^*)^2$$

On the lattice:

$$S_H[\phi] = \int_\Gamma d^dx (\Delta_\mu \phi)(\Delta_\mu \phi^*) + m^2 \phi\phi^* + g(\phi\phi^*)^2$$

For numerical calculations useful – action using dimensionless quantities

$$S_H[\phi] = \sum_n \left(-\kappa \sum_\mu \Re(\phi_n \phi^*_{n+e_\mu}) + \phi_n \phi^*_n + \lambda(\phi_n \phi^*_n - 1)^2 \right)$$

with

- $x_\mu = a n_\mu, \quad n_\mu \in \mathbb{Z}$
- $\phi_n = \frac{\sqrt{2} a^{d/2-1}}{\sqrt{\kappa}} \phi(na)$
- $1 - 2\lambda = \frac{1}{2}(2d + a^2 m^2)\kappa$
- $\kappa^2 = \frac{a^{d-4} 4\lambda}{g}$

So far, two coupling constants: κ, λ

What if $A_\mu \neq 0$?

Gauge Fields on the lattice

Remember: continuum

Let $\Lambda(x) = e^{-i\alpha(x)} \in \mathrm{U}(1)$: gauge group
$D(\Lambda(x)) = e^{-iQ\alpha(x)}$ a representation, charge $Q \in \mathbb{Z}$

Fields transform as:
$\phi(x) \to D(\Lambda(x))\phi(x)$
$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$

Usual problem: $\partial_\mu \phi$ not gauge invariant
solved with minimal substitution, i.e. $\partial_\mu \to D_\mu$

problems on the lattice:

- products like $\phi^*(y)\phi(x)$ are not gauge invariant
- how is the gauge field represented ?
- how to express the kinetic term $F_{\mu\nu}$?

solved by introducing transporters $U(s_{x,y})$

- allow comparison between ϕ at x and ϕ at y along curve s
- strict gauge invariance to all orders of a
- recover covariant derivative in continuum limit

Want special transformation behaviour

$$U(s_{x,y}) \to \Lambda(y)U(s_{x,y})\Lambda(x)^{-1}$$

Then products like

$$\phi^*(y)D(U(s_{x,y}))\phi(x)$$

are gauge invariant.

Gauge Fields on the lattice – cont.1

Solution: Define a transporter as Schwinger line integral

$$U(s_{x,y}) = e^{-ie\int_s A_\mu(z)dz_\mu}$$

Consider $s_{x,x+dx}$:

- gauge field A_μ $\rightarrow$ infinitesimal generator
- recover usual covariant derivative: $D_\mu = (\partial_\mu + ieQA_\mu(x))$

Gauge fields on the lattice

apply concept of transporters:

- elementary transporter $\rightarrow$ along a link (x,μ)

Higgs-part of action in gauge invariant manner:

$$S_H[\phi] = \sum_n \left(-\kappa \sum_\mu \Re(\phi^*_{n+e_\mu} D(U(s_{n,n+e_\mu}))\phi_n) + \phi_n\phi^*_n + \lambda(\phi_n\phi^*_n - 1)^2 \right)$$

Two common ways to define lattice gauge fields $\tilde{A}_\mu$, A'_μ

1. $U(s_{n,n+e_\mu}) = e^{-iea\tilde{A}_\mu(n)}$
2. $U(s_{n,n+e_\mu}) = 1 - ieaA'_\mu(n)$

Represent elementary transporter by compact link angles $\theta_{n,\mu} \in [-\pi,\pi)$:

$$U(s_{n,n+e_\mu}) = e^{-i\theta_{n,\mu}}$$

which live on the respective link. Thus

$$\theta_{n,\mu} = ae\tilde{A}_\mu(n)$$

$$\sin(\theta_{n,\mu}) = aeA'_\mu(n)$$

Gauge Fields on the lattice – cont.2

How represent kinetic term $\frac{1}{4}F_{\mu\nu}F_{\mu\nu}$?

Observation:

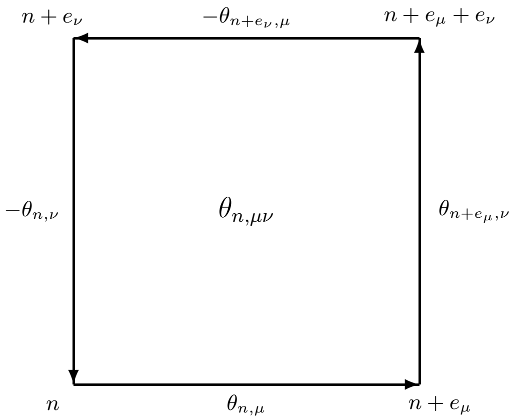
- product of elementary transporter gauge invariant if links form a closed loop
- smallest closed loop $\rightarrow$ plaquette

In analogy to the link angle define plaquette angle $\theta_{n,\mu\nu}$

$$e^{-i\theta_{n,\mu\nu}} := U_\nu^\dagger(n)U_\mu^\dagger(n+e_\nu)U_\nu(n+e_\mu)U_\mu(n)$$

with

$$\theta_{n,\mu\nu} = [\theta_{n,\mu} + \theta_{n+e_\mu,\nu} - \theta_{n,\nu} - \theta_{n+e_\nu,\mu}]_{2\pi}.$$



Easy to show

$$\theta_{n,\mu\nu} = ea^2 F_{\mu\nu}(n)$$

with lattice field strength tensor

$$F_{\mu\nu}(n) = \frac{1}{a}\left[(\tilde{A}_\nu(n+e_\mu) - \tilde{A}_\nu(n)) - (\tilde{A}_\mu(n+e_\nu) - \tilde{A}_\mu(n))\right]$$

Gauge Fields on the lattice – cont.3

Answer: Wilson action

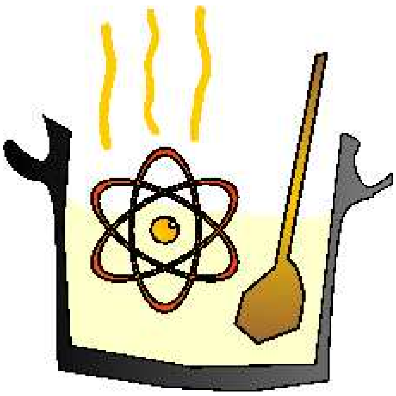
$$S_G[\theta] := \underbrace{\beta}_{\text{gauge coupling}} \sum_n \sum_{\mu<\nu} [1 - \cos\theta_{n,\mu\nu}]$$
$$= \beta \sum_P \left[1 - \cos\theta_P^{\text{plaq}}\right]$$

For $a \to 0$

- $\cos\theta_{n,\mu\nu} \to 1 - \frac{\theta^2_{n,\mu\nu}}{2}$
- Recover usual kinetic part of the action
- additional result

$$\beta = \frac{1}{e^2 a^{4-d}}, \text{ i.e. in 3 dimensions: } \beta \propto \frac{1}{a}$$

Now



Putting things together

Types of degrees of freedom

- Higgs field in polar coordinates $\phi_n = \rho_n e^{i\varphi_n}, \quad \varphi \in [-\pi, \pi)$
- Gauge field represented as link angles $\theta_{n,\mu}, \quad \theta_{n,\mu} \in [-\pi, \pi)$

Complete gauge invariante lattice action $S_{AHM} = S_H + S_G$:

$$\begin{aligned} S_{AHM} = &- \beta \sum_P \cos \theta_P^{\text{plaq}} \\ &- \kappa \sum_n \sum_\mu \rho_n \rho_{n+e_\mu} \cos(\varphi_{n+e_\mu} + Q\theta_{n,\mu} - \varphi_n) \\ &+ \sum_n (\rho_n^2 + \lambda(\rho_n^2 - 1)^2) \end{aligned}$$

Partition function:

$$Z_{AHM} = \int_{-\pi}^{\pi} \prod_n \frac{d\varphi_n}{2\pi} \int_0^{\infty} \prod_n \rho_n d\rho_n \int_{-\pi}^{\pi} \prod_{n,\mu} \frac{d\theta_{n,\mu}}{2\pi} \exp(-S_{AHM})$$

- $\lambda \to \infty$: London Limit, Higgs radial degrees of freedom frozen
  - $\beta \to \infty$: 3D-XY model
  - $\kappa \to \infty$: $\mathbb{Z}_Q$ gauge model
- $\beta \to \infty$: i.e. $a \to 0$ studying continuum limit
- $\kappa = 0$ or $Q = 0$: Decoupling of gauge and higgs field
  - pure gauge theory
  - ϕ^4 theory

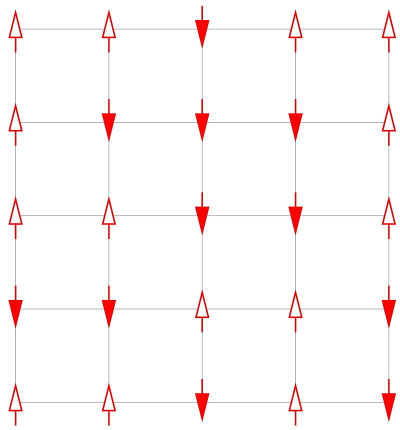
Most work done for $Q = 1$ and $\lambda \to \infty$

Statistical Physics

Given:

- Space C of configurations c_i (here assumed to be countable)
- probalitity distribution p over C
- Observables X over C

Expectation value of X given by $\langle X \rangle = \frac{\sum_{c \in \mathcal{C}} p(c) X(c)}{\sum_{c \in \mathcal{C}} p(c)}$



Complete enumeration
often not feasible

Monte Carlo Algorithm[a]:
A randomized algorithm that may produce incorrect results, but with bounded error probability.

Idea: Take random sample Γ and estimate the expectation value
Freedom: choose elements γ_j with distribution π from C

$$\langle X \rangle \approx \hat{X} := \frac{\sum_{\gamma \in \Gamma} \pi^{-1}(\gamma) p(\gamma) X(\gamma)}{\sum_{\gamma \in \Gamma} \pi^{-1}(\gamma) p(\gamma)}$$

Often used

- $\pi \equiv 1$ simple sampling
- $\pi \equiv p$ importance sampling

[a] From Algorithms and Theory of Computation Handbook, 15-21, Copyright (c) 1999 by CRC Press LLC

Markov processes

Importance sampling – Generate π by a Markov process!

Homogenian Markov process:

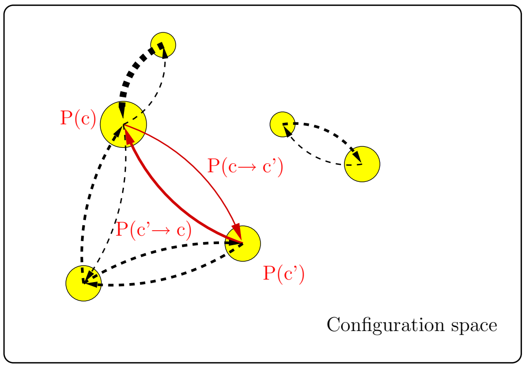
- stochastic process
- transition probability $P(c \to c')$ does only depend on former state c and not on time and earlier history

Master equation

$$P(c, t+1) - P(c, t) = \sum_{c'} [P(c', t)P(c' \to c) - P(c, t)P(c \to c')].$$

Static distribution $P(c, t) \to P_E(c)$ for each state c

$$\sum_{c'} [P_E(c')P(c' \to c) - P_E(c)P(c \to c')] = 0$$



To obtain given P_E require (sufficient):

1. Ergodicity
2. Detailed balance $P_E(c')P(c' \to c) - P_E(c)P(c \to c') = 0$

Remember *importance sampling*: $\pi \equiv P_E \stackrel{!}{=} p \propto e^{-S}$

Detailed balance gives

$$e^{-S(c)} P(c \to c') = e^{-S(c')} P(c' \to c)$$

Metropolis algorithm

Metropolis algorithm:

- simple, adaptable to many models
- efficient (if not too close to critical region)

$\longrightarrow$ extensively used in comp. physics

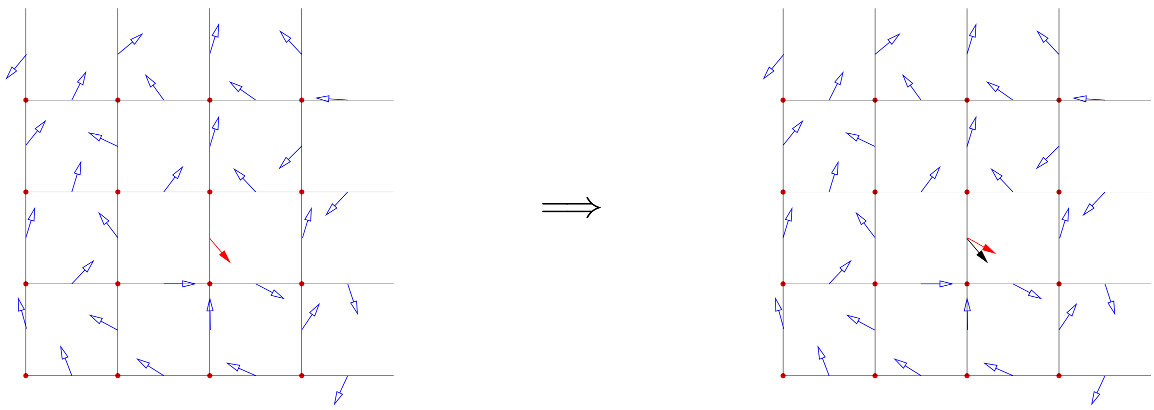
Consists of two steps

1. Proposal: Given c, propose c' with probability
   $P_0(c \to c') = P_0(c' \to c)$
2. Acceptance check: accept proposal with probability
   $P_1(c \to c') = \min[1, e^{S(c)-S(c')}]$

Local update: change only one degree of freedom f_i (e.g. $f_i = \phi_i, \theta_{i,\mu}$), while $c \to c'$

- action decomposes: $S(f_i) = S_{\rm loc}(f_i, U(i)) + \text{const}$
- local updates in typewriter fashion

In our case – Update of the gauge field with metropolis



$$\theta_{n,\mu} \to \theta'_{n,\mu} \equiv \theta_{n,\mu} + t\tau \quad \mod 2\pi$$

Heath-bath algorithm

Heat-bath algorithm:

- conceptionally simple, but often hard to implement
- usually outperforms Metropolis

Main idea:

$$P(c \to c') \propto e^{-S(c')}, \text{ or } P(f_i \to f_i') \propto e^{-S_{\mathrm{loc}}(f_i', U(i))}$$

$\to$ Problem of generating π is shifted to produce distribution for *one* degree of freedom using S_{loc}

Implementation:

Gaussian distribution can be generated efficiently (Box-Muller transformation)

$\to$ Combination from Metropolis and Heath-bath

Separate action $S_{\mathrm{loc}} = S_{\mathrm{Gaussian}} + S_{\mathrm{Remainder}}$

1. Generate new value from gaussian distribution $e^{-S_{\mathrm{Gaussian}}}$
2. Accept new value with probability $e^{-S_{\mathrm{Remainder}}}$

In particular:

$$S_{\mathrm{loc}}(\phi_n) = \underbrace{(\phi_n - b_n)^T(\phi_n - b_n)}_{\text{Gaussian}} + \underbrace{\lambda(\phi_n^T\phi_n - 1)^2}_{\text{Remainder}}$$

Additionally: Use mirror updates $\phi_n \to 2b_n - \phi_n$ in step 1

AHM – The Observables

Energetic observables:

- parts of the action
- *Higgs condensate* $\overline{\phi\phi^*} \to$ latent heat, interface tension

topological observables:

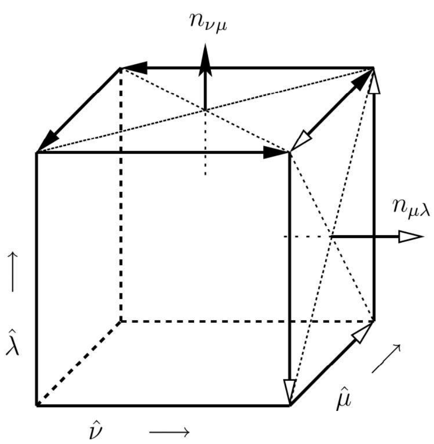
- compact gauge fields $\to$ topological defects = (anti-)monopoles, Dirac strings
- coupling with Higgs fields $\to$ ANO-strings

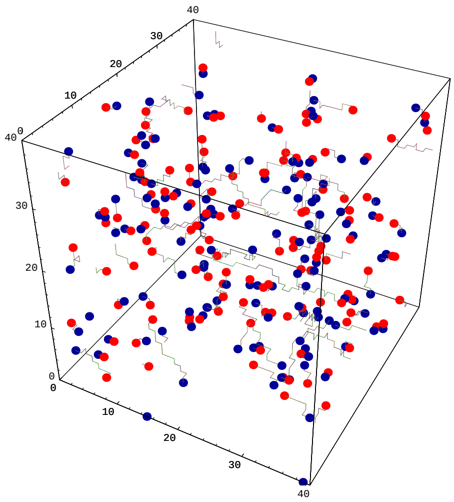
Some reasons:

- Villain approximation $\to$ partition function rewritten in terms of topological defects
- Believe: topological defects responsible for phase transition
- Percolation: defects form complicated clusters, study of its properties

Dirac strings $*s$, Monopoles $*j$ on the dual lattice[a]:

$$\theta_P = d\theta + 2\pi n \in [-\pi, \pi), \;\; j = dn$$



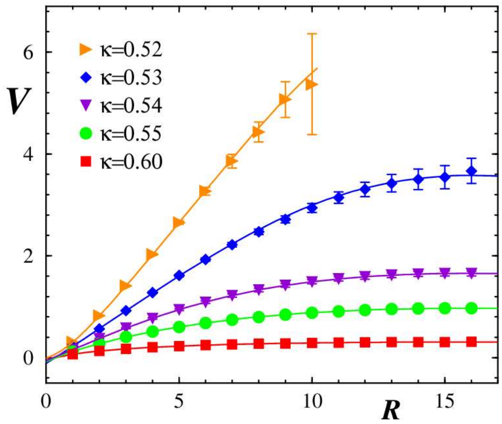


[a] left picture taken from [5]

AHM – The Phases

2 phases:

- symmetric-*confined* phase: monopole plasma $\longrightarrow$ confinement of test charges; realized for small couplings κ to the Higgs field
- Higgs-*deconfined* phase: linear potential suppressed; remaining monopoles bound into dipole pairs; larger κ



Potential V between two static external charges defined through Polyakov correlators[a]

$$\langle P(\vec{0}) P^*(\vec{(R)}) \rangle = e^{-LV(R)}$$

$$P(\vec{n} = (n_1, n_2, n_3)) = \exp\{i \sum_{n_3=1}^{L_3} \theta_{\vec{n},3}\}$$

[a] taken from [4]

Phase transition (for $Q = 1$)

- for small $\lambda \rightarrow$ 1^{st} order [7]
- large $\lambda \rightarrow$ crossover or 2^{nd} order or possibly Kosterlitz-Thouless type
- recent study at $\beta = 1.1$ [5]: investigation of interface tension using multicanonical updates

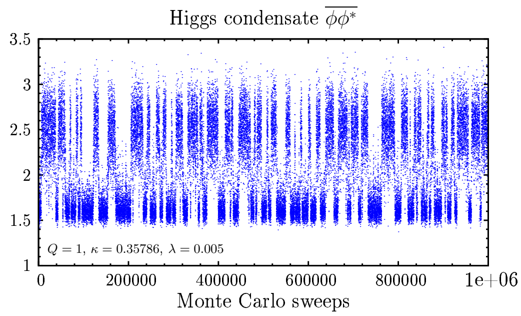
Phase transition (for $Q = 2$)

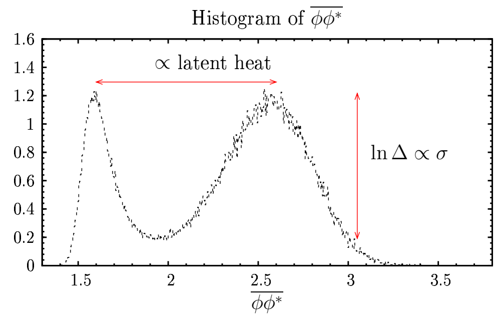
- 2^{nd} order for $\lambda \rightarrow \infty$ with non-universal critical exponents [6]

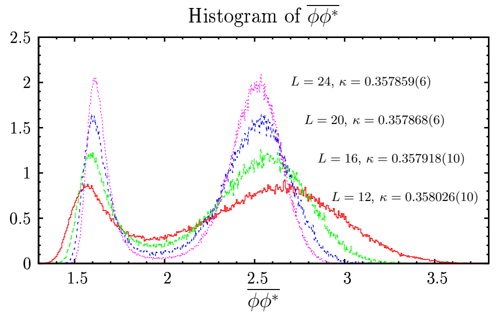
AHM – Types of phase transitions

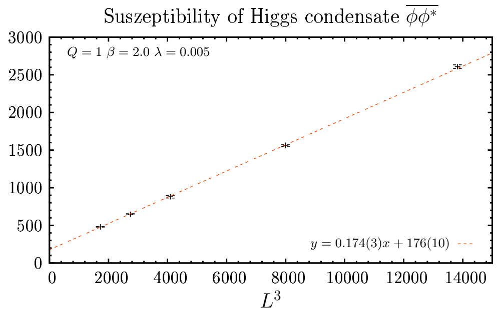
Ehrenfest: Transition of n-th order if some n-th partial derivative of some thermodynamical potential is not continuous

First order transition: Here mean action density not changing continuously with κ $\rightarrow$ look at higgs condensate (part of the action) – small λ

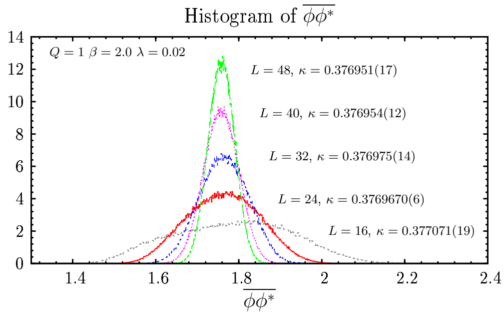


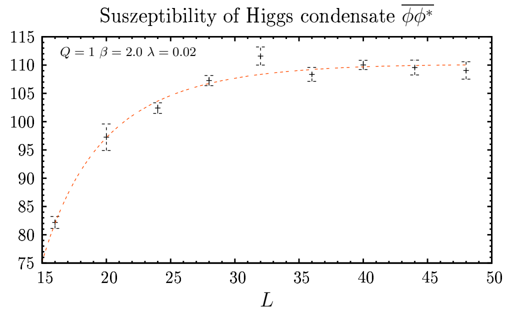






Continuous transition: continuous change of expectation value with external parameter – large λ





AHM – Phase structure

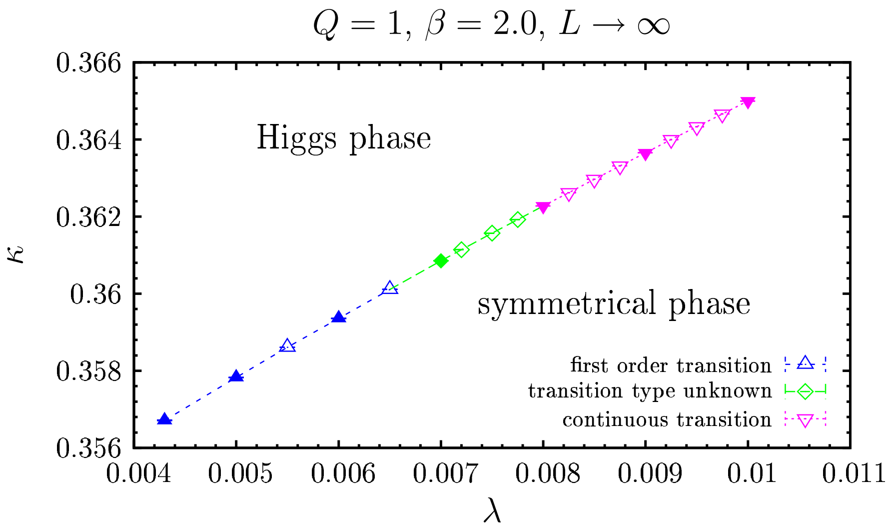
Numerical results

1. determine critical line in the λ, κ plane
2. determine type of phase transition
3. determine end point between transition type ($Q = 1$)
4. find appropriate parameter values for propagator measurements

How had this been achived?

- Do a lot of measurements in the critical region!
- To 1. Use reweighting techniques to extrapolate/interpolate to other parameter values
- To 2. Measure on different lattice sizes to allow for a finite size scaling analysis → critical exponents
  No critical behaviour for finite lattices
- To 3. Using interface tension, latent heat (obtained by histogram analysis), Binder cumulants...

The phase diagram:



The photon propagator

The dimensionless, gauge dependent propagator $D_{\mu\nu}$ defined as expectation value of gauge field correlations

$$D_{\mu\nu}(\vec{p}) = \langle \tilde{A}_\mu(\vec{k}) \tilde{A}_\nu(-\vec{k}) \rangle$$

where

$$\tilde{A}_\mu(\vec{k}) = \frac{1}{L^3} \sum_n e^{2\pi i \frac{1}{L} \sum_{\nu=1}^{3} k_\nu (n_\nu + \frac{1}{2}\delta_{\mu\nu})} A_{n+\frac{1}{2}e_\mu,\mu}$$

is just a Fourier transform of the gauge field $A_{n+\frac{1}{2}e_\mu,\mu} = \sin(\theta_{n,\mu})$ and lattice momenta $\vec{p}$ are given by

$$p_\mu(k_\mu) = \frac{2}{a} \sin\left(\frac{\pi k_\mu}{L_\mu}\right)$$

Assuming a rotational invariance and reality $\rightarrow$ Decomposition of the full propagator into transverse and longitudinal components

$$D_{\mu\nu}(\vec{p}) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) D(p^2) + \frac{p_\mu p_\nu}{p^2} \frac{F(p^2)}{p^2}$$

- Landau gauge fulfilled $\rightarrow$ $F(p^2)$ vanishes
- structure function $D(p^2)$ obtained via projection
- all information in $D(p^2)$

STOP! – Before you can measure the propagator $\rightarrow$ you have to fix the gauge

The Landau gauge

Here used: Landau gauge

Given by maximizing the gauge functional

$$\mathcal{G}[\alpha] = \sum_{n,\mu} \cos(\theta_{n,\mu} - \alpha_n + \alpha_{n+e_\mu})$$

necessary requirement

$$\frac{\delta \mathcal{G}[\alpha]}{\delta \alpha(n)} = 0$$

leads to

$$\sum_{\mu} [\sin(\theta_{n,\mu}) - \sin(\theta_{n-\mu,\mu})] = 0$$

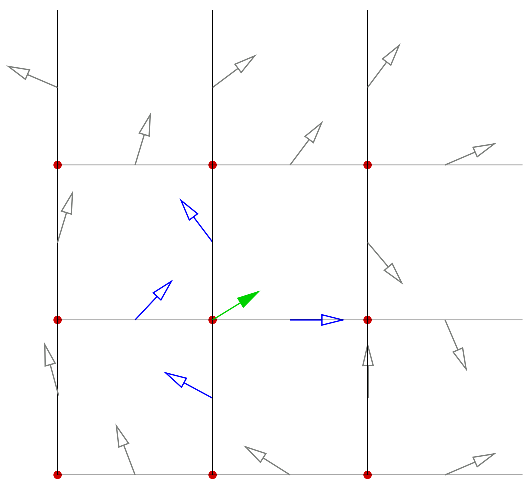
and gives you Landau/Lorentz gauge in continuum limit $\partial_\mu A_\mu = 0$.

How to maximize gauge functional?

Remember: gauge transformation corresponds to field α defined on lattice sites

$$\begin{aligned} \varphi_n \rightarrow & [\varphi_n - Q\alpha_n]_{2\pi} \\ \theta_{n,\mu} \rightarrow & [\theta_{n,\mu} - \alpha_n + \alpha_{n+e_\mu}]_{2\pi}. \end{aligned}$$

Idea: apply localized gauge transformations $\alpha^{(m)}$



$$\alpha_n^{(m)} = \omega \delta_{m,n}$$

with optimal ω

$$\tan(\omega) = \frac{\sum_\mu \left(\sin\theta_{n,\mu} - \sin\theta_{n-e_\mu,\mu}\right)}{\sum_\mu \left(\cos\theta_{n,\mu} + \cos\theta_{n-e_\mu,\mu}\right)}$$

The gauge fixing algorithm

Algorithm:

- Go through the whole lattice (*gauge fixing sweep*)
- For each lattice point m optimal (unique) ω can be determined
- Apply the localized gauge transformation and go to next lattice point
- Repeat with the next sweep until the quality of the fixing is good enough

Fundamental optimization problem: get stuck in local optimum

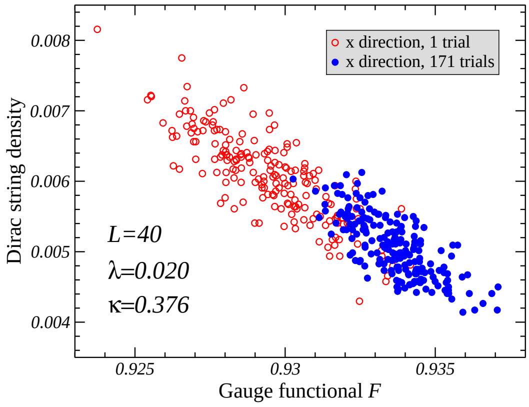
Approaches:

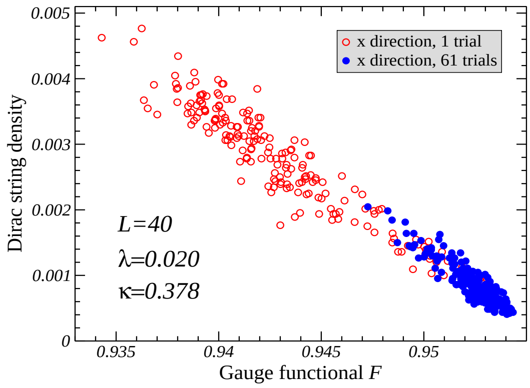
- several restarts from random gauge copies
- overrelaxation $\omega \rightarrow \eta\omega, \quad 1 \leq \eta \leq 2$, fastest convergence $\eta \approx 1.8$, $\eta = 2$ conserves gauge functional
- use other techniques like simulated annealing, ...

The gauge fixing algorithm – cont.

Locating global optimum much harder in the

confined-symmetric phase than in the deconfined-Higgs phase





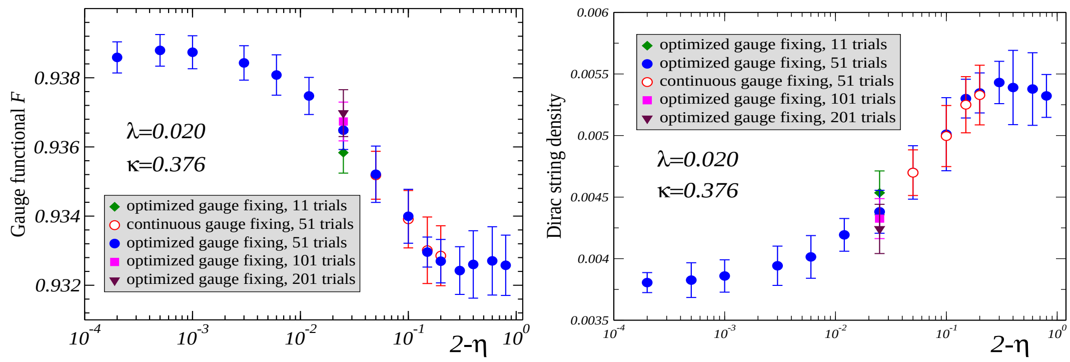
First Observation:

Maximizing gauge functional $\leftrightarrow$ *Minimizing* Dirac string density

- $\rightarrow$ intuitive understanding of difficulties
- (strong) local maximum = no Dirac loops left, remaining Dirac lines between monopoles minimize their length
- global maximum = certain pairing with minimal Dirac string density
- Higgs phase: diluted dipole gas $\rightarrow$ few pairings possible
- symmetric phase: dense monopole plasma $\rightarrow$ gauge fixing more demanding

Improving the algorithm

The role of the overrelation parameter



Second Observation:

Study of gauge functional and Dirac string density dependence on η

- better results for η close to 2
- better results if $\eta = 2$ and $\eta < 2$ steps applied in alternating order
- both methods are more effective than increasing number of restarts (regarding the number of total gauge fixing sweeps)

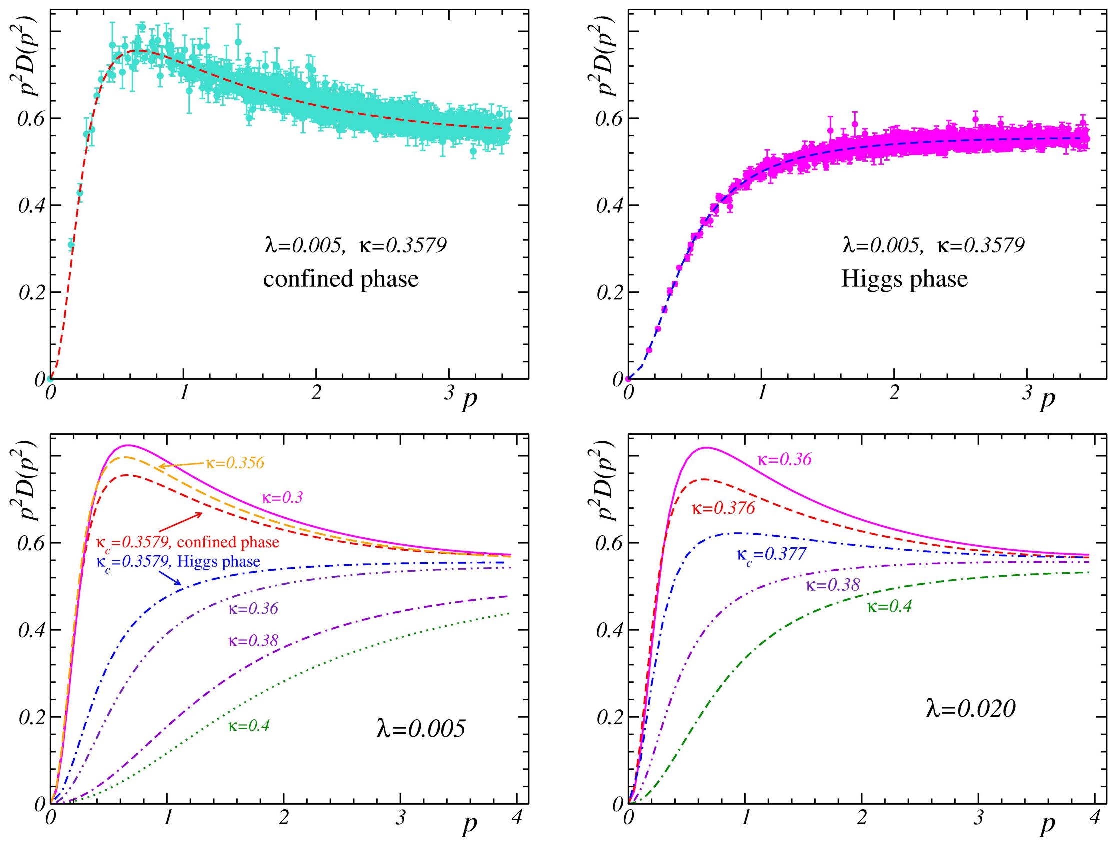
Improved Algorithm:

- use overrelaxation with η as high as computational time allows
- speed up the fixing by using
  - preselection: i.e. do several restarts but stop already if a weak limit is reached
  - use a discrete subgroup of U(1) at this stage (avoid trig. functions)
  - in a final stage use full U(1) with best candidate from preselection

The photon propagator – Results

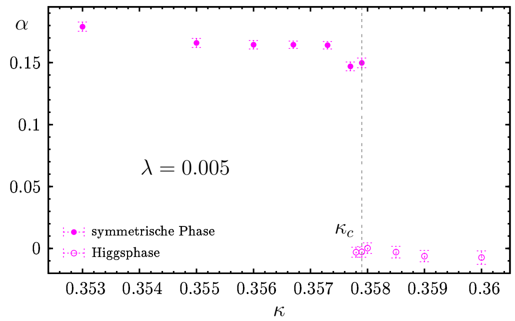
Ansatz: $$D(p^2) = \frac{Z m^{2\alpha}}{\beta(p^{2(1+\alpha)} + m^{2(1+\alpha)})} + C$$

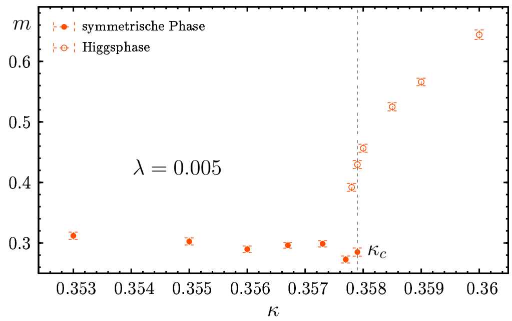
Momentum dependence of $p^2 D(p^2)$: $Q = 1$



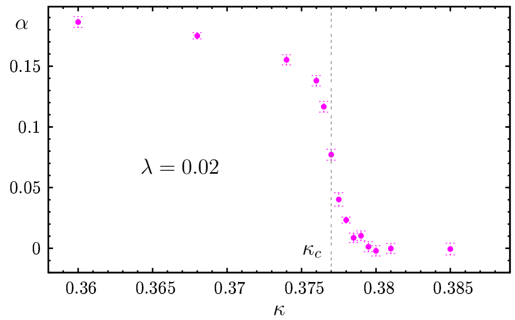
The photon propagator – Results cont.

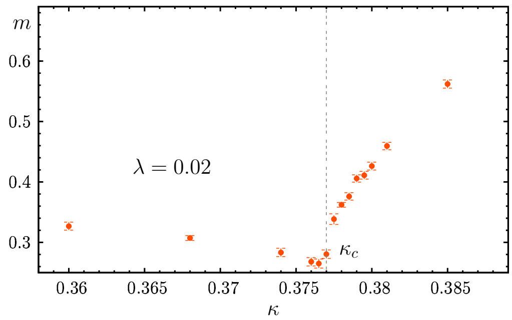
1^{st} order region – $Q = 1$



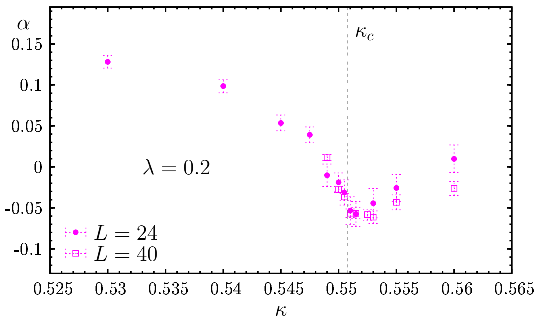


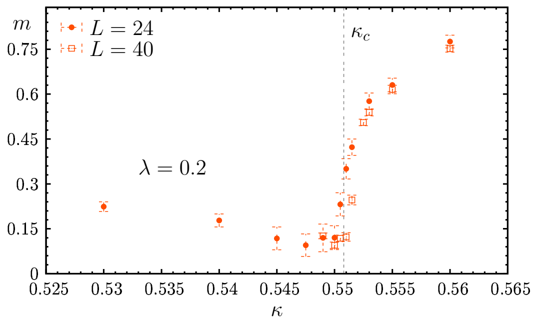
continuous transition region – $Q = 1$





2^{nd} order region – $Q = 2$





Summary

Summary

- Revised how:
  - The lattice AHM arises from a continuum scalar electrodynamics
  - Monte Carlo algorithms can be applied for numerical studies
- Phase structure qualitatively confirmed and quantitatively measured
- Improvement in the gauge fixing algorithm using preselection and discrete subgroups
- Propagator well described by 4 parameters: mass, anomalous dimension, renormalization constant, contact term
  - positive anomalous dimension α in confined phase
  - trivial massive gauge field propagator in the Higgs region
  - α strongly sensitive to successful gauge fixing, mass less sensitive
  - propagator reflects discontinuous change for small λ
  - continuous region: behaviour similar to London Limit (Schiller) $\rightarrow$ compact phase of the Higgs field is the main ingredient which influences the propagator

Currently under way – Study of cAHM$_3$ with $Q = 2$
new features:

- transition from first to second order
- monopoles can form chain like structures due to the double "valence" of the ANO strings
- propagators influenced by the topological objects

Thank you

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