On fermionic partition functions

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July 4, 2003

Abstract

This text originates from some theoretical work I did during May 2003 in order to prepare my diploma work. Its aim is to give the reader some insight into the changes arising if one deals with fermionic fields rather than bosonic ones. Usually fermionic fields differ in their field variables obeying anticommutation rules. Therefore one has to be careful about everyday operations like integration and differentiation. On the other hand an import theorem which states that taking the logarithm of the partition function removes all disconnected Feynman diagrams is recovered in the fermionic case. The reason is techniqually speaking that only elements of an even subalgebra of the considered Grassmann algebra are relevant. We will deal with this theorem and apply it to the special case of Yukawa coupling. The text relies for many reasons on [ms], however, I tried not to rephrase things but to add things left out and to translate them into a fermionic framework.

1 Introduction

Our starting point is the Euclidean action of a field theory, e.g. given by

$$S = \int dx \left[\phi(x)\frac{1}{2}(-\Delta + m_\phi^2)\phi(x)\right] + \lambda V(\phi)$$

The covariance C is defined by

$$\int dx\phi(x)(-\Delta + m_\phi^2)\phi(x) \equiv (\phi, C^{-1}\phi) \tag{1}$$

The bracket (.,.) denotes the scalar product, usually defined on a lattice. Here we shall not deal with the trouble involved going from the regularized lattice version into the continuum field theory. Therefore everything shall be understood in the "lattice sense", although we might use the continuum notation for convenience. The kernel of the operator $(-\Delta + m_\phi^2)^{-1}$ can be calculated easily since the laplacian gets diagonal in Fourier space.

$$C(x,y) = \int dp \frac{e^{ip(x-y)}}{p^2 + m_\phi^2}$$

The partition function $Z_\lambda(J)$ is defined as a functional integral over the field

$$Z_\lambda(J) = \mathcal{N}^{-1} \int D\phi e^{-S(\phi)+(J,\phi)} \tag{2}$$

$\mathcal{N}$ is an arbitrary but fixed normalisation constant and usually chosen to be $Z_0(0)$. From Z we can obtain the n-point correlation functions by differentiation with respect to the sources J.

$$\begin{aligned} G^{(n)}(x_1, \ldots, x_n) &\equiv \langle \phi(x_1) \cdot \ldots \cdot \phi(x_n) \rangle \\ &\equiv \left[\frac{\mathcal{N}}{Z_\lambda(0)} \frac{\delta^n}{\delta J(x_1) \ldots \delta J(x_n)} Z_\lambda(J) \right]_{J=0} \end{aligned}$$

Using (1) we can rewrite (2):

$$Z_\lambda(J) = \mathcal{N}^{-1} \int D\phi e^{-\frac{1}{2}(\phi, C^{-1}\phi)} e^{-\lambda V(\phi)+(J,\phi)} \tag{3}$$

This can be written as

$$Z_\lambda(J) = \int d\mu_C(\phi) e^{-\lambda V(\phi)+(J,\phi)} \tag{4}$$

Here we introduced the measure

$$d\mu_C(\phi) = (\det 2\pi C)^{-\frac{1}{2}} D\phi e^{-\frac{1}{2}(\phi, C^{-1}\phi)} \tag{5}$$

and we chose the normalisation constant $\mathcal{N}$ appropriately. For free fields (4) is a Gaussian integral and can be solved immediately

$$Z_0(J) = \int d\mu_C(\phi) e^{(J,\phi)} = e^{\frac{1}{2}(J,AJ)} \tag{6}$$

We will deal with the calculation of (4) for non-free fields in a pertubative manner. However, instead of evaluating $Z_\lambda(J)$ directly we shall do the calculation in a stepwise fashion and deal with the so-called effective action G_{eff}

$$G_{eff}(\chi, C, -\lambda V) = \log \int d\mu_C(\phi) e^{-\lambda V(\phi+\chi)}$$

It can be written in a series expansion in λ

$$\begin{aligned} G_{eff}(\chi, C, -\lambda V) &= \sum_{r=1}^{\infty} \lambda^r G_{eff}^r(\chi, C, V) \\ &= \sum_{r=1}^{\infty} \lambda^r \left[\frac{\partial^r}{\partial \lambda^r} \log \int d\mu_C(\phi) e^{-\lambda V(\phi+\chi)} \right]_{\lambda=0} \end{aligned} \tag{7}$$

2 Grassmann Algebra

The set of objects $\psi_i, \quad i \in \{1, \dots, N\}$ is called a set of Grassmann variables if the objects obey the following *anticommutation* rule

$$\{\psi_i, \psi_j\} = 0 \quad \forall i, j \in \{1, \dots, N\} \tag{8}$$

On important consequence is

$$\psi_i^2 = 0 \tag{9}$$

The set of all polynomials in the variables with complex coefficients together with addition, multiplication and scalar multiplication constitutes the Grassmann Algebra $\mathcal{A}[\psi]$. From (9) is follows immediately that the grassmann algebra is finite with 2^N elements. A basis for $\mathcal{A}$ is given by

$$B = \{1\} \cup \{\psi_{i_1} \cdot \dots \cdot \psi_{i_n} \mid n \geq 1, i_1, \dots, i_n \in \{1, \dots, N\} \quad i_1 < i_2 < \dots < i_n\}$$

Functions on the grassmann algebra are defined on their power series expansion. Differentiation is defined as a linear map $\frac{\partial}{\partial \psi_i} : \mathcal{A} \to \mathcal{A}$ with

$$\frac{\partial}{\partial \psi_i} \psi_{i_1} \dots \psi_{i_n} = \sum_{k=1}^{n} (-1)^{k-1} \delta_{i, i_k} \psi_{i_1} \dots \psi_{i_k - 1} \psi_{i_k + 1} \dots \psi_{i_n} \tag{10}$$

Since elements of the Grassmann algebra are at most linear in a given variable it suffices to define the integration for linear and constant functions.

$$\begin{aligned} \int d\psi_i &= 0 \\ \int d\psi_i \psi_i &= 1 \end{aligned} \tag{11}$$

As well as $\{\psi_i | i \in \{1, \dots, N\}\}$, $\{\frac{\partial}{\partial \psi_i} | i \in \{1, \dots, N\}\}$ and $\{d\psi_i | i \in \{1, \dots, N\}\}$ are generators of a Grassmann algebra. Due to (10) and (11) differentiation and integration with respect to a Grassmann variable are essentially the same.

3 Wick ordering and Measure

We start with some definitions and results for the bosonic case. Wick ordering Ω_C with respect to the covariance C is a linear map defined on the algebra of polynomials in the field variables. It is defined on the monomials as follows:

$$\begin{aligned} \Omega_C(1) &= 1 \\ \Omega_C(\phi(x_1) \dots \phi(x_n)) &= \left[\left(\prod_{k=1}^{n} \frac{1}{i} \frac{\partial}{\partial \alpha_k} \right) e^{i(\alpha, \phi) + \frac{1}{2}(\alpha, C\alpha)} \right]_{J=0} \end{aligned} \tag{12}$$

Due to the commutation of the derivation Ω_C is permutation invariant and therefore well-defined. According to its linearity the Wick ordering $\Omega_C(e^{i(\alpha,\phi)})$ shall be given as formal power series in α, i.e.

$$\Omega_C(e^{i(\alpha,\phi)}) = \sum_{n=0}^{\infty} \frac{i^n}{n!}\Omega((\alpha,\phi)^n) = e^{i(\alpha,\phi)+\frac{1}{2}(\alpha,C\alpha)} \tag{13}$$

The proof of the last equality can be found in [ms]. Now let us consider fermions. In our formalism two sets of Grassmann variables $\{\bar{\psi}_i\},\{\psi_i\}$ will be used, which together form a Grassmann algebra. Now lets define a Wick ordering in the case of a fermionic field with the variables $\{\bar{\psi}_i\},\{\psi_i\}$. The use of labeling the coordinate variables will become clear later on. The idea is to start with the demand that $\Omega_C(ab) = ab - \langle ab\rangle$. Everything further should then follow from the analogy to (12). Hence we define:

$$\Omega_C(\prod_{i=1}^{\tilde{m}_k}\psi(x_{k,i})\prod_{i=\tilde{m}_k+1}^{m_k}\bar{\psi}(x_{k,i})) \equiv$$

$$\left[\left(\prod_{i=\tilde{m}_k+1}^{m_k}\frac{1}{i}\frac{\partial}{\partial\eta_{k,i}}\prod_{i=1}^{\tilde{m}_k}\frac{1}{i}\frac{\partial}{\partial\bar{\eta}_{k,i}}\right)e^{i(\bar{\psi},\eta^{(k)})+i(\bar{\eta}^{(k)},\psi)+(\bar{\eta}^{(k)},C\eta^{(k)})}\right]_{\bar{\eta},\eta=0} \tag{14}$$

Where $\eta^{(k)}(x)$ (and analogous $\bar{\eta}^{(k)}$,$\alpha^{(k)}$) is defined as

$$\eta^{(k)}(x) \equiv \sum_{i=1}^{\tilde{m}_k}\eta_{k,i}\delta(x,x_{k,i})$$

The delta function can be understood in the lattice sense as a Kronecker delta. The anticommutation rule of the derivatives make the fermionic Wick ordering antisymmetric. Please convince yourself, that (14) leads to an identity if we use a covariance C=0 and use $\Omega_0(f(\psi,\bar{\psi})) = f(\psi,\bar{\psi})$. The measure in the fermionic case is given by

$$d\mu_C(\bar{\psi},\psi) = \frac{1}{\det(C^{-1})}D\bar{\psi}D\psi e^{-(\bar{\psi},C^{-1}\psi)} \tag{15}$$

Apart from the det factor in front and the factor $\frac{1}{2}$ in the exponential (15) looks the same as (5). The analog to (6) is the following

$$\int d\mu_C(\bar{\psi},\psi)e^{(\bar{\eta},\psi)+(\bar{\psi},\eta)} = e^{(\bar{\eta},C\eta)}$$

Of course these notions can be generalised to a system of arbitrarily many fermionic and bosonic fields. This is simply done by factorizing the Wick ordering and the measure in products of fermionic and bosonic Wick orderings and measures. We shall use the shorthand $\Omega_{\vec{C}}(\ldots)$ and $d\mu_{\vec{C}}(\ldots)$ whenever appropriate. The Gaussian integral of the system therefore decomposes into usual bosonic and fermionic Gaussian integrals. We shall use a generalisation of (13)

$$\Omega_C(e^{i(\bar{\psi},\eta)+i(\bar{\eta},\psi)}) = e^{i(\bar{\psi},\eta)+i(\bar{\eta},\psi)+(\bar{\eta},C\eta)} \tag{16}$$

4 Calculation of the Effective Action

In this section we want to derive an expression for the effective action. The Feynman rules will then be found in the next paragraph. The fundamental result is that only connected graphs contribute to G_{eff}. Disconnected graphs are therefore removed by taking the logarithm of the partition function. This is proven in the bosonic case and we will find that this result is recovered in fermionic and bosonic/fermionic systems. In order to avoid some index-related confusion we shall restrict our consideration to the case of a system with just one fermionic and bosonic field. The generalisation to arbitrary many fields is then straightforward.

Lets us start with (7). The interactions shall be given as *even* in the fermionic fields, i.e $V_k(\bar{\psi},\psi,\phi)$ and $V_i(\bar{\psi},\psi,\phi)$ shall commute. Further we assume that every interaction can be written as a polynomial in the fields, i.e.

$$\begin{aligned} V_k(\bar{\psi},\psi,\phi) &= \sum_{m_k=1}^{M_k} \int \prod_{i=1}^{m_k} dx_{k,i} v_k^{(m_k)}(x_{k,1},\dots,x_{k,m_k}) \cdot \\ &\cdot \Omega_{C^B}(\prod_{i=1}^{m_k^B} \phi(x_{k,i}))\Omega_{C^F}(\prod_{i=m_k^B+1}^{\tilde{m}_k} \psi(x_{k,i}) \prod_{i=\tilde{m}_k+1}^{m_k} \bar{\psi}(x_{k,i})) \end{aligned} \tag{17}$$

Now we use (16),(13) and the definitions of Ω_{C^B} and Ω_{C^F} in order to write

$$\begin{aligned} V_k(\bar{\psi},\psi,\phi) &= \sum_{m_k=1}^{M_k} \int \prod_{i=1}^{m_k} dx_{k,i} v_k^{(m_k)}(x_{k,1},\dots,x_{k,m_k}) \cdot \\ &\cdot \left[\frac{\partial^{m_k^B}}{\partial i\alpha_{k,1}\dots\partial i\alpha_{k,m_k^B}} \Omega_{C^B}(e^{i(\alpha^{(k)},\phi)}) \right]_{\alpha=0} \end{aligned} \tag{18}$$
$$\cdot \left[\frac{\partial^{m_k-m_k^B}}{\partial i\eta_{k,\tilde{m}_k+1}\dots\partial i\eta_{k,m_k}\partial i\bar{\eta}_{k,m_k^B+1}\dots\partial i\bar{\eta}_{k,\tilde{m}_k}} \Omega_{C^F}(e^{i(\bar{\psi},\eta^{(k)})+i(\bar{\eta}^{(k)},\psi)}) \right]_{\bar{\eta},\eta=0}$$

Using the following identity

$$\left[(\frac{\partial}{\partial\lambda})^n f(\lambda x) \right]_{\lambda=0} = \left[\frac{\partial^n}{\partial\lambda_1\dots\partial\lambda_n} f(\lambda_1 x_1 \dots \lambda_n x_n) \right]_{\lambda_i=0,x_i=x}$$

we can write the expression $G_{eff}^{(r)}(\chi,\bar{\xi},\xi,\vec{C}_1,V) \equiv \mathcal{E}(\underbrace{V,\dots,V}_{r-times};\vec{C}_1,\chi,\bar{\xi},\xi)$ as

$$\begin{aligned} &\mathcal{E}(V_1,\dots,V_r;\vec{C}_1,\chi,\bar{\xi},\xi) = \\ &\quad \left[\frac{\partial^r}{\partial\lambda_1\dots\partial\lambda_r} \log \int d\mu_{C_1}^B(\phi) d\mu_{C_1}^F(\bar{\psi},\psi) e^{-\sum_{i=1}^r \lambda_i V_i(\bar{\psi}+\bar{\xi},\psi+\xi,\phi+\chi)} \right]_{\lambda_i=0} \end{aligned} \tag{19}$$

With (19) and using the fact that $\mathcal{E}(V_1,\dots V_r;C_1,\chi,\bar{\xi},\xi)$ is linear in every interaction V_k we obtain

$$\mathcal{E}(V_1,\dots,V_r;\vec{C}_1,\chi,\bar{\xi},\xi)=\sum_{m_1\dots m_r}\int\prod_{k=1}^{r}\prod_{i=1}^{m_k}dx_{k,i}\prod_{k=1}^{r}v_k^{(m_k)}(x_{k,1},\dots,x_{k,m_k})$$
$$\left[\prod_{k=1}^{r}\prod_{i=1}^{m_k^B}\frac{\partial}{\partial i\alpha_{k,i}}\prod_{k=1}^{r}\left(\prod_{i=\tilde{m}_k+1}^{m_k}\frac{\partial}{\partial i\eta_{k,i}}\prod_{i=m_k^B+1}^{\tilde{m}_k}\frac{\partial}{\partial i\bar{\eta}_{k,i}}\right)E(\alpha,\bar{\eta},\eta,\vec{C},\vec{C}_1,\chi,\bar{\xi},\xi)\right]_{\alpha,\bar{\eta},\eta=0} \quad (20)$$

Our purpose is hence to calculate

$$E(\alpha,\bar{\eta},\eta,\vec{C},\vec{C}_1,\chi,\bar{\xi},\xi)=\mathcal{E}(\Omega_{C^B}(e^{i(\alpha^{(1)},.)})\Omega_{C^F}(e^{i(.,\eta^{(1)})+i(\bar{\eta}^{(1)},.)}),\dots;\vec{C}_1,\chi,\bar{\xi},\xi) \quad (21)$$

We shall not go through the details but merely try to find the changes induced by the fermionic fields. We can rearrange (21) into

$$E=\left[\frac{\partial^r}{\partial\lambda_1\dots\partial\lambda_r}\log Z\right]_{\lambda_i=0} \quad (22)$$

with

$$Z=\int d\mu_{\vec{C}_1}(\phi,\bar{\psi},\psi)\prod_{k=1}^{r}(1+\lambda_k\Omega_{C^B}(e^{i(\alpha^{(k)},\phi+\chi)})\Omega_{C^F}(e^{i(\bar{\psi}+\bar{\xi},\eta^{(k)})+i(\bar{\eta}^{(k)},\psi+\xi)}))$$

Expanding the product and deploying (13), (16) leads now to a sum of Gaussian integrals.

$$\begin{aligned} Z &= \sum_{M\subset\{1,\dots,r\}}\int d\mu_{\vec{C}_1}(\phi,\bar{\psi},\psi) \\ &\quad \prod_{k\in M}e^{i(\alpha^{(k)},\phi+\chi)+\frac{1}{2}(\alpha^{(k)},C^B\alpha^{(k)})}e^{i(\bar{\psi}+\bar{\xi},\eta^{(k)})+i(\bar{\eta}^{(k)},\psi+\xi)+(\bar{\eta}^{(k)},C^F\eta^{(k)})} \\ &= \sum_{M\subset\{1,\dots,r\}}(\prod_{m\in M}\lambda_m\zeta_m)(\prod_{\substack{k,l\in M\\ k<l}}\beta_{kl;\vec{C}_1}) \end{aligned} \quad (23)$$

Where

$$\begin{aligned} \zeta_m &\equiv e^{i(\alpha^{(k)},\chi)+\frac{1}{2}(\alpha^{(k)},C_2^B\alpha^{(k)})+i(\bar{\xi},\eta^{(k)})+i(\bar{\eta}^{(k)},\xi)+(\bar{\eta}^{(k)},C_2^F\eta^{(k)})} \\ \beta_{kl;\vec{C}} &\equiv e^{-(\alpha^{(k)},C^B\alpha^{(l)})-(\bar{\eta}^{(k)},C^F\eta^{(l)})-(\bar{\eta}^{(l)},C^F\eta^{(k)})} \end{aligned} \quad (24)$$

and we made use of the covariance splitting $C=C_1+C_2$. Now we see that every term in (23) is bilinear and therefore in the even subalgebra of the Grassmann algebra. Therefore we can now proceed as in the usual bosonic case without bothering about fermions. We will not repeat this calculation here but rather sketch the way it is done. Starting from the expression of Z given in (23) we transform it into a *polymer* partition function. There we can use a theorem stating that the logarithm of Z is given by a sum over connected graphs. Finally we are only interested (according to (22)) in taking the derivatives of $logZ$ at the point $\lambda_i = 0$. This simplifies the result considerably and we end up with

$$E(\alpha, \bar{\eta}, \eta, \vec{C}, \vec{C}_1, \chi, \bar{\xi}, \xi) = (\prod_{k=1}^{r} \zeta_k) \sum_{G \in \mathcal{G}_c(\{1,\ldots,r\})} \prod_{(i,j)\in G} (\beta_{ij;\vec{C}_1} - 1) \tag{25}$$

For iteration of the semigroup of the effective action we just need to Wick order the result with respect to C_2, i.e. with respect to the field variables not yet integrated over. Using (13) and (16) we obtain

$$\prod_{k=1}^{r} \zeta_k = \Omega^B_{C_2}(e^{i\sum_{k=1}^r (\alpha^{(k)},\chi)}) \Omega^F_{C_2}(e^{i\sum_{k=1}^r (\bar{\xi},\eta^{(k)}) + i\sum_{k=1}^r (\bar{\eta}^{(k)},\xi)}) \prod_{\substack{k,l=1 \\ k<l}}^{r} \beta_{kl;\vec{C}_2}$$

Since $\prod \beta = \prod(1 + (\beta - 1))$ we find that (25) can also be written as

$$\begin{aligned} &E(\alpha, \bar{\eta}, \eta, \vec{C}, \vec{C}_1, \chi, \bar{\xi}, \xi) = \\ &\quad \Omega^B_{C_2}(e^{i\sum_{k=1}^r (\alpha^{(k)},\chi)}) \Omega^F_{C_2}(e^{i\sum_{k=1}^r (\bar{\xi},\eta^{(k)}) + i\sum_{k=1}^r (\bar{\eta}^{(k)},\xi)}) \\ &\quad \sum_{G \in \mathcal{G}(\{1,\ldots,r\})} \prod_{(i,j)\in G} (\beta_{ij;\vec{C}_2} - 1) \sum_{G \in \mathcal{G}_c(\{1,\ldots,r\})} \prod_{(i,j)\in G} (\beta_{ij;\vec{C}_1} - 1) \end{aligned} \tag{26}$$

Here $\mathcal{G}_{(c)}(\{1,\ldots,r\})$ denotes the set of all (connected) graphs having vertices $\{1,\ldots,r\}$.

5 Feynman rules

Our starting points in this section are the equations (25) and (20). We will make use of the following Lemma (it is stated without proof in [ms]) in order to express $\mathcal{E}$ in terms of Feynman diagrams.

Lemma 1 *Let A and B elements of a Grassmann algebra $\mathcal{A}[\eta]$, i.e.*

$$A = \sum_I a_I \eta^I, \quad B = \sum_I b_I \eta^I$$

with complex coefficients a_I and b_I and let C be a formal power series in λ, i.e.

$$C(\lambda) = \sum_{r=0}^{\infty} c_r \lambda^r$$

Then

$$\left[A(\frac{\partial}{\partial\eta}) B(\eta) C((\eta,\psi)) \right]_{\eta=0} = \left[A(\eta) B(\frac{\overleftarrow{\partial}}{\partial\eta}) C((\frac{\overleftarrow{\partial}}{\partial\eta},\psi)) \right]_{\eta=0} \tag{27}$$

Proof. We expand everything into products of monomials of A,B and C. Then we show that the relation holds for monomials. After this we combine the monomials to polynoms A,B and C again. Let us therefore consider without loss in generality the following case

$$\begin{aligned} c = c_r(\eta,\psi)^r &= c_r \sum_{k_1} \dots \sum_{k_r} \eta_{k_1}\psi_{k_1} \dots \eta_{k_r}\psi_{k_r} \\ &= c_r \sum_{k_1} \dots \sum_{k_r} (-1)^{\frac{r(r-1)}{2}} \eta_{k_1} \dots \eta_{k_r}\psi_{k_1} \dots \psi_{k_r} \end{aligned}$$

Using monomials of A, B and C we obtain

$$\left[a_I \frac{\partial}{\partial \eta^I} b_J \eta^J \eta_{k_1} \dots \eta_{k_r} \psi_{k_1} \dots \psi_{k_r} \right]_{\eta=0} = sign(\pi) a_I b_J \psi_{k_1} \dots \psi_{k_r}$$

where the permutation π maps the indices $I = (i_1, \dots, i_{l+r})$ onto the indices $(J,K) = (j_1, \dots, j_l, k_1, \dots, k_r)$ with

$$\pi(i_{l+r}) = j_1 \dots \pi(i_{r+1}) = j_l, \quad \pi(i_r) = k_1 \dots \pi(i_1) = r_r$$

On the other hand

$$\left[a_I \eta^I b_J \frac{\overleftarrow{\partial}}{\partial \eta^I} \frac{\overleftarrow{\partial}}{\partial \eta_{k_1}} \dots \frac{\overleftarrow{\partial}}{\partial \eta_{k_r}} \psi_{k_1} \dots \psi_{k_r} \right]_{\eta=0} = sign(\pi) a_I b_J \psi_{k_1} \dots \psi_{k_r}$$

Remark. Due to the bilinearity of C, we can interchange the order of B and C if needed.

Applying this Lemma to (25) and (20) we obtain the following rules for calculating $\mathcal{E}$:

- Think of sources $\alpha, \bar{\eta}, \eta$ connected with every leg
- Build the following graphs
  - remove sources and add the corresponding external fields to chosen legs
  - chose pairs from the remaining legs and connect them pairwise with a line (propagator) of covariance C_1
    make sure that the the graph is now connected by these lines
  - chose pairs from the remaining legs and connect them with a line of covariance C_2
    The graph may be disconnected if C_2 lines are considered alone
  - Set the sources to zero, .i.e. if there are still legs with sources on it, the value of the graph is zero.
- Multiply all propagators, vertex functions and fields and integrate over all coordinate variables and sum over all possible graphs.

A bit more formally we get

Theorem 1

$$\mathcal{E}(V_1,\ldots,V_r;\vec{C}_1,\chi,\bar{\xi},\xi) =$$
$$\sum_{m_1,\ldots,m_r}\sum_{m,k,l=0}\int dy_1\ldots dy_m dz_1\ldots dz_k d\bar{z}_1\ldots d\bar{z}_l$$
$$\Omega_{C_2^B}(\chi(y_1)\ldots\chi(y_m))\Omega_{C_2^F}(\xi(z_1)\ldots\xi(z_k)\bar{\xi}(\bar{z}_1)\ldots\bar{\xi}(\bar{z}_l))$$
$$\sum_{E\subset B}\sum_{L\in L_c(B_0\backslash E)}\sum_{h:L\to\{1,2\}} Val_{G_L^{(h)}}(\vec{C}_1,\vec{C}_2,y,z,\bar{z})$$

Where the value of a Feynman graph is defined as

$$Val_{G_L^{(h)}}(\vec{C}_1,\vec{C}_2,y,z,\bar{z}) =$$
$$\int\prod_{b\in B_0\backslash E} dx_b \prod_{k=1}^{r} v_k^{(m_k)}(x_{k,1}\ldots x_{k,m_k})\prod_{l=(b,b')\in L}\tilde{C}_{h(l)}(x_b,x_{b'})$$

if the Graph L is connected by C_1 propagators alone.
Otherwise $Val_{G_L^{(h)}}(\vec{C}_1,\vec{C}_2,y,z,\bar{z}) = 0.$
With

$$B_0 = \{(k,i)|k\in\{1,\ldots,r\},\quad i\in\{1,\ldots,m_k\}\}$$
$$E = (\{b_1,\ldots,b_m\},\{c_1,\ldots,c_k\},\{\bar{c}_1,\ldots,\bar{c}_l\})\quad \textit{where}\quad y_u = x_{b_u},\quad z_v = x_{c_v},\quad \bar{z}_w = x_{\bar{c}_w}$$

$$\tilde{C}_{h(l)}(x_{k,i},x_{k',i'}) \equiv$$
$$\begin{cases} C^B_{h(l)}(x_{k,i},x_{k',i'}) & if\quad 1\le i,i'\le m_k^B \\ C^F_{h(l)}(x_{k,i},x_{k',i'}) & if\quad m_k^B+1\le i\le \tilde{m}_k,\quad \tilde{m}_k+1\le i'\le m_k \\ C^F_{h(l)}(x_{k',i'},x_{k,i}) & if\quad \tilde{m}_k+1\le i\le m_k,\quad m_k^B+1\le i'\le\tilde{m}_k \\ 0 & otherwise\end{cases}$$

6 Yukawa coupling - An example

Although the theorem stated above may look rather tricky its meaning is actually very simple as shown in the "rules" above. We will now apply our knowledge to the special case of Yukawa coupling. The Euclidian action is given here as

$$\begin{aligned} S &= \int dx\left[\phi(x)\frac{1}{2}(-\Delta+m_\phi^2)\phi(x)+\bar{\psi}(x)(i\gamma^\mu\partial_\mu-m_\psi)\psi(x)\right]+\lambda\int dx\phi(x)\psi(x)\bar{\psi}(x) \\ &= \frac{1}{2}(\phi,C_B^{-1}\phi)+(\bar{\psi},C_F^{-1}\psi)+\lambda\int dx\phi(x)\psi(x)\bar{\psi}(x)\end{aligned}$$

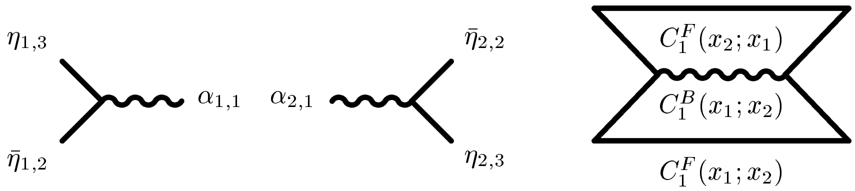
Since we would like to deal with Wick ordered interactions we get according to (17)

$$
\begin{aligned}
V_k(\bar{\psi}, \psi, \phi) &= \int \prod_{i=1}^{3} dx_{k,i} \delta(x_{k,1} - x_{k,2}) \delta(x_{k,2} - x_{k,3}) \cdot \\
&\cdot \Omega_{C^B}(\phi(x_{k,1})) \Omega_{C^F}(\psi(x_{k,2}) \bar{\psi}(x_{k,3}))
\end{aligned}
$$

From the above presented theorem it follows that

$$
\begin{aligned}
&\mathrm{Val}^{Yukawa}_{G_L^{(h)}}(\vec{C}_1, \vec{C}_2, y, z, \bar{z}) \\
&= \int \prod_{k=1}^{r} dx_k \prod_{l=(k,i;k',i') \in L} \tilde{C}_{h(l)}(x_k, x_{k'}) \quad \text{if connected by } C_1 \text{ lines alone} \\
&= 0 \quad \text{otherwise}
\end{aligned}
$$

In the following figure a second order diagram for the Yukawa theory is shown in the special case of no external fields and $C_2 = 0$. On the left each leg is connected with sources, in the right picture the sources are removed and the legs are pairwise connected by C_1 propagators.



What about the symmetry factor in this theory? Two labelled Feynman diagrams are called topologic equivalent if they can be transformed into another by reordering equivalent legs of vertices (no exchange between different vertices allowed!). In the Yukawa case at each vertex there are 3 legs of different kinds (in terms of the sources they are α, $\bar{\eta}$ and η legs) and they are cleary not equivalent. Thus since topological equivalence implies identity in this particular case the symmetry factor, which represents the number of topologic equivalent Feynman diagrams, equals one. This is a very simple and rather peculiar result.

7 Conclusion

We see that handling fermionic fields needs some care. On the other hand straightforward extensions of the bosonic notions including the use of Grassman variables do their job. In the worked out framework for handling fermions it is now easy to deal with bosonic/fermionic systems of arbitrary degree (in the number of fields). The effective action can be expressed in terms of sums over Feynman graphs in a well-defined way. In particular we notice that disconnected diagrams are removed just as in the bosonic case.

References

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