THE SECRET LIFE OF SOLITONS

These lecture notes were prepared for the Mitteldeutsche Physik Combo at Leipzig and Jena in June and July 2010. They describe the connection between classical soliton equations and the quantum field theory of free fermions in 1+1 dimensions. This remarkable and unexpected relationship was uncovered in the early 1980’s by the Kyoto school of Mikio Sato.

In preparing the notes, I have relied heavily on the monograph Soliton Equations and Hamiltonian Systems, by Leonid A. Dickey, and to lesser extent on the book by three of Sato’s collaborators: Solitons: Differential equations, symmetries and infinite dimensional algebras, by T. Miwa, M. Jimbo, and E. Date. The only part for which I claim any originality is in my account of the Fermi-Bose equivalence. For this, see the introductory material in Bosonization, M. Stone ed. (WSPC 1994).

The Korteweg-de-Vries equation

The first soliton was observed by John Scott Russell on the Union Canal, near Hermiston in Scotland. Finite-amplitude waves in shallow water are well described by the Korteweg-de-Vries (KdV) equation

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{\partial^3 u}{\partial x^3},
\]

and Russell’s “great wave of translation” is its hump-like solitary wave solution

\[
u(x,t) = 3\alpha^2 \text{sech}^2 \left\{ \frac{1}{2} (\alpha x - \alpha^3 t) \right\}.\]

The hump travels at speed \(\alpha^2\), so the larger the amplitude, the faster it moves. The KdV equation also possesses many-solitary-wave solutions in which larger and faster humps overtake and pass through slower humps, each surviving unchanged expect for a phase shift. This particle-like soliton property is quite remarkable. What makes the solitary waves so robust?

The three coefficients before the terms in the KdV equation can be changed by rescalings \(q \to \alpha q, x \to \beta x, t \to \gamma t\) by constants \(\alpha, \beta\) and \(\gamma\) to any chosen set. From the version

\[ (\partial_t + u \partial_x)u = \delta \partial_{xxx}^3 u, \]
we can deduce conservation laws

\[
\begin{align*}
\partial_t u + \partial_x \left\{ \frac{1}{2} u^2 - \delta \partial_{xx}^2 u \right\} &= 0, \\
\partial_t \left\{ \frac{1}{2} u^2 \right\} + \partial_x \left\{ \frac{1}{3} u^3 - \delta u \partial_{xx}^2 u + \frac{1}{2} \delta (\partial_x u)^2 \right\} &= 0 \\
& \vdots
\end{align*}
\]

where the dots indicate that we can keep going and find infinitely many conserved quantities. When \( \delta = 0 \) these reduce to the equations

\[
\partial_t \left\{ \frac{1}{n} u^n \right\} + \partial_x \left\{ \frac{1}{n+1} u^{n+1} \right\} = 0
\]

indicating that \( \int_{-\infty}^{\infty} u^n dx \) is time independent. The \( \delta = 0 \) limit is sick however, and some modification, such as KdV’s \( u''' \) term, is needed to keep the solutions from becoming multivalued.

Now most modifications to \((\partial_t + u \partial_x) u = 0\) destroy all but one conservation law. For example Burgers’ equation

\[
(\partial_t + u \partial_x) u = \eta \partial_{xx}^3 u,
\]

leads to

\[
\partial_t u + \partial_x \left\{ \frac{1}{2} u^2 - \eta \partial_x^2 u \right\} = 0.
\]

This tells us that \( \int u \, dx \) is still conserved — but no other conservation law survives once \( \eta \) becomes non-zero.

It is the infinite family of conservation laws that protect the solitons. What is special about the innocent-looking KdV equation that grants it such an abundance of conserved quantities?

**Lax Pair**

The Schrödinger operator

\[
L = -\partial_x^2 + q(x)
\]

is formally self adjoint, \( L^\dagger = L \), with respect to the inner product

\[
\langle u, v \rangle = \int_{-\infty}^{\infty} u^* v \, dx
\]
The third-order operator
\[ P = \partial_x^3 + a(x)\partial_x + \partial_x a(x) \]
is skew-adjoint, \( P^\dagger = -P \), with respect to this inner product. In \( P \) the combination \( \partial_x a(x) \) means “first multiply by \( a(x) \), and then differentiate the result,” so we could also write
\[ \partial_x a = a\partial_x + a'. \]
Now form the commutator \([P, L] \equiv PL - LP\). After a little effort, we find
\[ [P, L] = (3q' + 4a')\partial_x^2 + (3q'' + 4a'')\partial_x + q''' + 2aq' + a''. \]
If we choose \( a = -\frac{3}{4}q \), the commutator becomes a pure multiplication operator, with no derivative operator:
\[ [P, L] = \frac{1}{4}q''' - \frac{3}{2}qq'. \]
The equation
\[ \frac{dL}{dt} = [P, L], \]
can therefore be written as
\[ \frac{\partial q}{\partial t} = \frac{1}{4} \frac{\partial^3 q}{\partial x^3} - \frac{3}{2} \frac{\partial q}{\partial x}, \]
which is a version of KdV. The operators \( P \) and \( L \) are called Lax pair, after Peter Lax who uncovered much of the structure.

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**Aside 1:** Many other soliton-possessing equations can be written in the Lax form. For example let
\[ L = \begin{bmatrix} i\partial_x & \chi^* \\ \chi & i\partial_x \end{bmatrix}, \quad P = \begin{bmatrix} i|\chi|^2 & -\chi''* \\ \chi' & -i|\chi|^2 \end{bmatrix}, \]
then
\[ \frac{dL}{dt} = [L, P] \]
is equivalent to the pair of equations
\[
\dot{\chi}^* = -i\chi''^* - 2i\chi^*|\chi|^2; \\
\dot{\chi} = +i\chi'' + 2i\chi|\chi|^2,
\]
or, equivalently
\[
-i\dot{\chi}^* = -\chi''^* - 2\chi^*|\chi|^2; \\
i\dot{\chi} = -\chi'' - 2\chi|\chi|^2.
\]
Of course only one of these is needed, as the other is its complex conjugate. This is the \textit{non-linear Schrödinger equation}. It describes the motion of a condensate of Bose particles with attractive interactions, and also the evolution of the envelope of waves in materials with a non-linear refractive index.

Another example takes
\[
L = \partial^3_x + u\partial_x + v \\
P = \partial^2_x + q.
\]
Then \([P, L]\) becomes
\[
(2u' - 3q')\partial^2_x + (u'' + 2v' - 3q'')\partial_x + (v'' - uu' - q''').
\]
If we set \(q = 2/3u\), then the coefficient of \(\partial^2_x\) vanishes and it becomes consistent to set \(dL/dt = [P, L]\), whence
\[
\dot{u} = -u'' + 2v', \\
\dot{v} = v'' - \frac{2}{3}u''' - \frac{2}{3}uu'.
\]
Eliminating \(v\) gives
\[
\ddot{u} = -\frac{1}{3}u^{(4)} - \frac{2}{3}(u^2)'''.
\]

This is the \textit{Boussinesq equation}.

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The existence of an infinite number of conservation laws for the KdV equation is related to its Lax-pair form. From
\[
\frac{dL}{dt} = [P(t), L(t)],
\]

we find a formal solution

$$L(t) = U(t)L(0)U^{-1}(t),$$

where $U(t)$ is the (formally) unitary operator found by solving the equation

$$\frac{dU}{dt} = P(t)U(t).$$

Now $P(t)$ depends on $q(x,t)$ and hence on $L$, so this is really quite a complicated equation — but the key fact is that the time evolution of $L$ is being given by a unitary transformation, and a unitary transformation does not change the eigenvalues of $L$: if

$$L(0)\psi_0(x) = \lambda\psi_0(x),$$

then $\psi_t = U(t)\psi_0$ obeys

$$L(t)\psi_t(x) = \lambda\psi_t(x).$$

Perhaps these eigenvalues are related to the conserved quantities?

The energy eigenvalues of a Schrödinger operator are complicated functionals of the potential $q(x)$. It is however possible to find combinations of the eigenvalues that are easier to compute, and these combinations should still be time independent. Let us seek them.

A tool for accessing the eigenvalue spectrum of $L$ is provided by the associated \textit{resolvent}. This is the operator

$$R(\lambda) = (L - \lambda)^{-1}.$$ 

It is a kind of Green function for $L$, but with an extra parameter $\lambda$. If the operator $L$ has only a discrete spectrum, and we know the eigenvalues $\lambda_n$ and the eigenfunctions $\psi_n(x) \equiv \langle x | n \rangle$, then

$$R(x,y;\lambda) = \langle x | (L - \lambda)^{-1} | y \rangle = \sum_n \langle x | n \rangle \frac{1}{\lambda_n - \lambda} \langle n | y \rangle.$$ 

We see that $R(x,y;\lambda)$ has poles at the location of the eigenvalues. When the spectrum has a continuous part, the corresponding poles will merge to form a branch cut. The advantage of using the resolvent is that, like all Green functions, the resolvent is \textit{local}. For $x, y$ in the vicinity of some point,
$R(x, y; \lambda)$ can be computed in terms of the coefficients in the differential operator near that point. In particular, there is a not exactly obvious (but easy to obtain once you know the trick) local gradient expansion for the diagonal elements $R(x) \equiv R(x; x; \lambda)$. These elements are all that we will need. We begin by recalling that we construct the Green function by setting

$$R(x, y; \lambda) \propto u(x)v(y)$$

where $u(x)$, $v(x)$ are solutions of $(-\partial_x^2 + q(x) - \lambda)y = 0$ satisfying suitable boundary conditions to the right and left respectively. We set $R(x) = R(x, x; \lambda)$ and differentiate three times with respect to $x$. We find

$$\partial_x^3 R(x) = u^{(3)}v + 3u''v' + 3u'v'' + uv^{(3)} = (\partial_x(q - \lambda)u) v + 3(q - \lambda)\partial_x(uv) + (\partial_x(q - \lambda)v) u.$$ 

Here, in passing from the first to second line, we have used the differential equation obeyed by $u$ and $v$. We can re-express the second line as

$$(q\partial_x + \partial_x q - \frac{1}{2}\partial_x^3)R(x) = 2\lambda\partial_x R(x).$$

This relation is known as the Gelfand-Dikii equation. Using it we can find an expansion for the diagonal element $R(x)$ in terms of $q$ and its derivatives. We begin by observing that for $q(x) \equiv 0$ we know that $R(x) = 1/(2\sqrt{-\lambda})$. We therefore conjecture that we can expand

$$R(x; \lambda) = \frac{1}{2\sqrt{-\lambda}} \left( 1 + \frac{b_1(x)}{2\lambda} + \frac{b_2(x)}{(2\lambda)^2} + \cdots + \frac{b_n(x)}{(2\lambda)^n} + \cdots \right).$$

If we insert this expansion into Gelfand-Dikii we see that we get the recurrence relation

$$(q\partial_x + \partial_x q - \frac{1}{2}\partial_x^3)b_n = \partial_x b_{n+1}.$$ 

We can therefore find $b_{n+1}$ from $b_n$ by differentiation followed by a single integration. Remarkably, $\partial_x b_{n+1}$ is always the exact derivative of a polynomial in $q$ and its derivatives. Further, the integration constants must be be zero so that we recover the $q \equiv 0$ result. A simple Mathematica program will generate

$$b_1(x) = q(x),$$

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and so on. (Note how the terms in the expansion are graded: Each $b_n$ is homogeneous in powers of $q$ and its derivatives, provided we count two $x$ derivatives as being worth one $q(x)$.) Keeping a few terms in this series expansion can provide an effective approximation for $R(x)$, but, in general, the series is not convergent, being only an asymptotic expansion.

For a potential $q(x)$ possessing a discrete spectrum case we would expect that

$$
\int_{-\infty}^{\infty} R(x) \, dx = \sum_n \frac{1}{\lambda_n - \lambda} \geq -\frac{1}{\lambda} \left( 1 + \frac{1}{\lambda} \sum \lambda_n + \frac{1}{\lambda^2} \sum \lambda_n^2 + \ldots \right),
$$

so the integrals of the $b_n(x)$ should be related to the sums of powers of the eigenvalues. Unfortunately the eigenvalues can be unboundedly large so these sums cannot possibly converge. This divergence is reflected in our expansion of $R(x, x)$ involving half integer powers of $\lambda$, rather than integer powers. The $\int b_n \, dx$ can instead be related to the analytic continuation of the convergent zeta-function sums

$$
\zeta_L(s) = \sum_n \left( \frac{1}{\lambda_n^s} - \frac{1}{\lambda_{n,0}^s} \right) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left[ \int_{-\infty}^{\infty} \left\{ \langle x | e^{-tL} | x \rangle - \frac{1}{\sqrt{4\pi t}} \right\} \, dx \right] \, dt
$$

to negative powers of $s$. We find that $\zeta_L(s)$ has poles at $s = (1/2 - n)$ whose residue is proportional to the $\int b_n \, dx$.

Because they are functions only of the eigenvalues, we conjecture that the $\int b_n(x) \, dx$ will be conserved quantities. Soon we will see that this conjecture is true.

**KdV Hierarchy**

The commutator of the differential operator $P$ with $L$ contained no differential operators. Let’s see if we can find other operators with the same
property. An obvious first one is simply $D = \partial_x$, since

$$[D, L] = q'.$$

We follow J. L. Burchnall, T. W. Chaundy, Proc. Lond. Math. Soc 21 (1923) 420, and seek operators $Q_{2n-1}$ such that $[Q_{2n-1}, L] = u'_n$. Here $u_n$ is to be some differential polynomial in $u_1(x) \equiv q(x) = b_1(x)$. In particular we already have $Q_1 = D$ (and $Q_3 \propto P$). The idea is that, given a $Q_{2n-1}$ and its $u_n$, we can construct $Q_{2n+1}$ and $u_{n+1}$ recursively. Set

$$R = Q_{2n-1}L + LQ_{2n-1},$$

then we have

$$[R, L] = u'_nL + Lu'_n = u'_n(-D^2 + u_1) + (-D^2 + u_1)u'_n = -2u'_nD^2 - 2u''_nD - u^{(3)}_n + 2u_1u_n.$$

Combine $R$ with

$$\frac{1}{2}(u_nD + Du_n) = u_nD + \frac{1}{2}u'_n$$

which obeys,

$$[u_nD + \frac{1}{2}u'_n, L] = 2u'_nD^2 + 2u''_nD + \frac{1}{2}u^{(3)}_n + u_nu'_1$$

and we find that

$$[Q_{2n-1}L + LQ_{2n-1} + \frac{1}{2}(u_nD + Du_n), L] = -\frac{1}{2}u^{(3)}_n + 2u_1u'_n + u_nu'_1.$$

Thus

$$u'_{n+1} = -\frac{1}{2}u^{(3)}_n + 2u_1u'_n + u_nu'_1 = (u_1D + Du_1 - \frac{1}{2}D^3)u_n.$$

and

$$Q_{2n+1} = Q_{2n-1}L + LQ_{2n-1} + \frac{1}{2}(u_nD + Du_n).$$

The recurrence and initial conditions for $u_n(x)$ coincide with that for the Gelfand-Dikii coefficients $b_n(x)$!
We can write
\[
Q_{2n+1} = Q_{2n-1}L + LQ_{2n-1} + \frac{1}{2}(u_nD + Du_n)
\]
\[
= Q_{2n-1}2L + [L, Q_{2n-1}] + \frac{1}{2}(u_nD + Du_n)
\]
\[
= Q_{2n-1}2L + u_nD - \frac{1}{2}u'_n
\]

If we set \(u_0(x) \equiv 1\) and recall that \(Q_1 = D\), we have
\[
Q_3 = (u_0D - \frac{1}{2}u'_0)(2L) + u_1D - \frac{1}{2}u'_1
\]
\[
Q_5 = (u_0D - \frac{1}{2}u'_0)(2L)^2 + (u_1D - \frac{1}{2}u'_1)(2L) + (u_2D - \frac{1}{2}u'_2)
\]
and in general
\[
Q_{2n+1} = \sum_{k=0}^{n} (u_kD - \frac{1}{2}u'_k)(2L)^{n-k}.
\]

We could also define \(Q_{2n} = (2L)^n\), but now the commutator is trivially zero.

We use these new operators to generate a family of time evolutions called the KdV Hierarchy:
\[
\frac{dL}{dt_{2n-1}} = [Q_{2n-1}, L].
\]
Equivalently
\[
\frac{dq}{dt_{2n-1}} = \partial_x b_n[(x)].
\]
Also
\[
\frac{dL}{dt_{2n}} = 0.
\]

Each of the time evolutions provides an isospectral deformation of the Schrödinger potential \(q(x)\). What we don’t see yet is that the evolutions commute. We can evolve from one set of multi-times \((t_1, t_2, \ldots)\) to another \((t'_1, t'_2, \ldots)\), and it does not matter which \(t_n\) we vary first. This means that the operator \(L\) is actually a function of the \(t_n\).

Hamiltonians

One way of seeing that flows commute, and that the \(\int b_n(x)\, dx\) are conserved quantities, is to show that the KdV Hierarchy is generated by a set of Poisson-commuting Hamiltonians.
A useful tool here is the Heat kernel associated with $L$

$$G(x, y; t) = \langle x|e^{-tL}|y \rangle = \sum_n \langle x|n \rangle e^{-t\lambda_n} \langle n|y \rangle.$$ 

It is simply the Laplace transform of the resolvent

$$\int_0^\infty G(x, y; t)e^{t\lambda} dt = R(x, y; \lambda),$$

and so contains the same information. In particular our expansion and

$$R(x, x; \lambda) = \frac{1}{2\sqrt{-\lambda}} \left( 1 + \frac{b_1(x)}{2\lambda} + \frac{b_2(x)}{(2\lambda)^2} + \cdots + \frac{b_n(x)}{(2\lambda)^n} + \cdots \right)$$

yields a short-time expansion

$$G(x, x; t) = \frac{1}{\sqrt{4\pi t}} \left( 1 - b_1(x)t + \frac{1}{3}b_2(x)t^2 - \frac{1}{3\cdot5}b_3(x)t^3 + \cdots \right).$$

We can use it to find another relation between adjacent $b_n(x)$.

We begin by constructing the functional

$$E[q] = \int_{-\infty}^\infty G(x, x, t) \, dx = \sum_n e^{-\lambda_n t}.$$ 

Make a variation $q(x) \rightarrow q(x) + \delta q(x)$, then

$$\delta E = \sum_n \left\{ -t\delta \lambda_n e^{-\lambda_n t} \right\},$$

$$= -t \sum_n \langle n|\delta q|n \rangle e^{-\lambda_n t}, \quad (1\text{-st order perturbation theory})$$

$$= -t \int_{-\infty}^\infty G(x, x, t)\delta q(x) \, dx.$$ 

Comparing coefficients of $t^{n-1/2}$ on the two sides tells us that

$$\frac{\delta}{\delta q(x)} \int b_n(y) \, dy = (2n - 1)b_{n-1}(x).$$
For example
\[
\frac{\delta}{\delta q(x)} \int b_3(y) \, dy = \frac{\delta}{\delta q(x)} \int \left\{ \frac{5}{2} q^3 - \frac{5}{4} (q')^2 - \frac{5}{2} q q'' + \frac{1}{4} q^{(4)} \right\} \, dy
\]
\[
= \frac{\delta}{\delta q(x)} \int \left\{ \frac{5}{2} q^2 + \frac{5}{4} (q')^2 \right\} \, dy \quad \text{(discarding total derivatives)},
\]
\[
= 5 \left\{ \frac{3}{2} q^2 - \frac{1}{2} q'' \right\}
\]
\[
= 5 b_2(x)
\]

Now we follow H. McKean, P. van Moerbeke, Inv. Math 30 (1975) 217, and define the Hamiltonian functionals
\[
H_n[q] = \frac{1}{2n + 1} \int b_{n+1}(x) \, dx.
\]
Then the KdV Hierarchy equations can be written in terms of their functional derivatives:
\[
\frac{\partial}{\partial x} \left( \frac{\delta H_n}{\delta q(x)} \right) = \partial_x b_n(x) = \frac{dq}{dt_{2n-1}}
\]
If \( F[q] \) is a functional of \( q \), we therefore have
\[
\frac{dF}{dt_{2n-1}} = \int \frac{\delta F}{\delta q(x)} \left( \frac{dq}{dt_{2n-1}} \right) \, dx = \int \frac{\delta F}{\delta q(x)} \frac{\partial}{\partial x} \left( \frac{\delta H_n}{\delta q(x)} \right) \, dx
\]
This result suggests that we define a *Poisson Bracket*
\[
\{ F[q], G[q] \} = \int \frac{\delta F}{\delta q(x)} \frac{\partial}{\partial x} \left( \frac{\delta G}{\delta q(x)} \right) \, dx
\]
for any pair of functionals \( F \) and \( G \). Now we can write
\[
\frac{dF}{dt_{2n-1}} = \{ F[q], H_n[q] \}.
\]
This bracket symbol a genuine Poisson bracket in that it satisfies the Jacobi identity:
\[
\{ F, \{ G, H \} \} + \{ G, \{ H, F \} \} + \{ H, \{ F, G \} \} = 0.
\]
(Reader: Check this!) When Hamiltonians \( H_1 \) and \( H_2 \) give rise to phase-space flows \( \mathbf{v}_{H_1} \) and \( \mathbf{v}_{H_2} \), then the Jacobi identity tells us that
\[
[\mathbf{v}_{H_1}, \mathbf{v}_{H_2}] = \mathbf{v}_{\{ H_1, H_2 \}}
\]
where \([u, v]\) denotes the Lie bracket of vector fields. This means that when we have a set of Hamiltonians \(H_n\) that have the property that each is conserved by the flows generated by all the others, \(i.e.\) when

\[
\frac{dH_n}{dt_{2m-1}} = \{H_n, H_m\} = 0, \quad \text{for all } n, m,
\]

then these flows \textit{commute} because their mutual Lie brackets vanish.

We now show that the KdV hierarchy \(H_n\) are indeed all conserved: We have

\[
\frac{d}{dt_{2m-1}} \int b_n \, dx = \int \left( \frac{\delta}{\delta q(x)} \int b_n(y) \, dy \right) \frac{dq}{dt_{2m-1}} \, dx
\]

\[
= (2n - 1) \int b_{n-1}(\partial_x b_m) \, dx
\]

\[
= -(2n - 1) \int \partial_x(b_{n-1})b_m \, dx
\]

\[
= -(2n - 1) \int \left\{ \left( qD + Dq - \frac{1}{3}D^3 \right) b_{n-2} \right\} b_m \, dx
\]

\[
= (2n - 1) \int b_{n-2} \left\{ \left( qD + Dq - \frac{1}{3}D^3 \right) b_m \right\} \, dx
\]

\[
= (2n - 1) \int b_{n-2}(\partial_x b_{m+1}) \, dx
\]

\[
= \frac{2n - 1}{2n - 3 \, dt_{2m+1}} \int b_{n-1} \, dx
\]

But

\[
\frac{d}{dt_1} \int b_n \, dx = 0
\]

for all \(n\), as the flow is just translation. From this we conclude that the KdV hamiltonians are mutually Poisson commuting:

\[
\{H_n[q], H_m[q]\} = 0, \quad \forall n, m \geq 0.
\]

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\textbf{Aside 2:} We can also define a second Poisson Bracket

\[
\{F[q], G[q]\}_2 = \int \frac{\delta F}{\delta q} \left( q \frac{\partial}{\partial x} + \frac{\partial}{\partial x} q - \frac{1}{3} \frac{\partial^3}{\partial x^3} \right) \frac{\delta G}{\delta q} \, dx
\]
The first bracket gives rise to a classical version of a Kac-Moody current algebra and the second to a classical version of the Virasoro algebra. The two brackets are related by the Kupershmidt-Wilson theorem: Suppose that $u = q^2 + q'$, then a computation will show that

$$\{F[u(q)], G[u(q)]\}_q = 2\{F[u], G[u]\}_{2,u}.$$ 

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Pseudodifferential Operators and the Volterra Group

We now present a rather different approach to the KdV hierarchy. It will provide an explanation for some of the magic.

From

$$Df = fD + f'$$

we are motivated to set

$$D^{-1}DfD^{-1} = D^{-1}fDD^{-1} + D^{-1}fD^{-1}$$

or

$$D^{-1}f = fD^{-1} - D^{-1}f'D^{-1}. \quad (\star)$$

Now $D$ is not really an invertible operator, but we will take $(\star)$ to be the defining property of the symbol $D^{-1}$. Continuing in this vein, we have

$$D^{-1}f = fD^{-1} - D^{-1}f'D^{-1}$$

$$= fD^{-1} - f'D^{-2} + D^{-1}f''D^{-2}$$

$$= fD^{-1} - f'D^{-2} + f''D^{-3} - D^{-1}f^{(3)}D^{-4},$$

and ultimately to the formal sum

$$D^{-1}f = \sum_{k=0}^{\infty} (-1)^k f^{(k)} D^{-k-1}.$$ 

We are not concerned with convergence: a pseudo-differential operator

$$P = \sum_{k=0}^{\infty} p_k(x) D^{m-k}$$

is simply a convenient way of keeping track of and manipulating the sequence of functions $p_k(x)$. 

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A useful formula is
\[
(f D^m)(g D^n) = \sum_{k=0}^{\infty} \frac{1}{k!} : \partial^k_x (f D^m) \partial^k_x (g D^n) :
\]
Here \( \ldots : \) indicates a normal ordering in which \( D \) goes to the right of all functions. For \( n, m \) positive, upper the limit on the sum can be replaced by \( m \), and
\[
\sum_{k=0}^{m} \frac{1}{k!} : \partial^k_x (f D^m) \partial^k_x (g D^n) : = \sum_{k=0}^{m} \left( \begin{array}{c} n \\ k \end{array} \right) f g^{(k)} D^{n+m-k},
\]
but the series also works when either \( n \) or \( m \) is negative. The effect is to make the space of pseudodifferential operators into an associative ring.

In the ring, operators of the form \( X = D^n + a_1 D^{n-1} + \cdots \) possess an inverse \( D^{-n} + b_1 D^{-n-1} + \cdots \). For example
\[
(D^2 - q)^{-1} = D^{-2} + qD^{-4} - 2q'D^{-5} + \cdots
\]
The inverse makes the set of operators of the form
\[
1 + a_1 D^{-1} + a_2 D^{-2} + \cdots
\]
into a group – the Volterra group. The group has a Lie algebra that consists of elements
\[
b_1 D^{-1} + b_2 D^{-2} + \cdots.
\]
Also operators of the form
\[
D^n + \text{lower}
\]
possess unique \( n \)-th roots of the form
\[
D + \text{lower}.
\]
For example
\[
(D^2 - q)^{1/2} = D - \frac{1}{2} (qD - \frac{1}{2} q') (D^2 - q)^{-1} + \\
\quad + \frac{1}{4} \left( \left( \frac{3}{2} q^2 - \frac{1}{2} q'' \right) D - \frac{1}{2} (3qq' - \frac{1}{2} q^{(3)}) \right) (D^2 - q)^{-2} + \ldots
\]
\[
= D + \frac{1}{2} q D^{-1} - \frac{1}{4} q' D^{-2} + \frac{1}{8} (q'' - q^2) D^{-3} + \frac{1}{6} (6qq' - q^{(3)}) D^{-4} + \ldots
\]
14
We recognize here terms appearing in the operator $Q_{2n+1}$. We have that

$$Q_{2n+1} = (L^{(2n+1)/2})_+,$$

where $(\ldots)_+$ means discard the terms containing negative powers of $D$. The KdV hierarchy can therefore be written as

$$\frac{dL}{dt_n} \equiv \frac{\partial q}{\partial t_n} = [(L^{n/2})_+, L].$$

Granted that we know how to compute the roots, we can see why things had to work out this way. Consider

$$X = aD^n + \text{lower}$$

and

$$Y = bD^m + \text{lower}$$

then

$$[X, Y] = (nab' - m'a'b)D^{n+m-1} + \text{lower}$$

So if $L = D^2 - q$ we have that

$$[(L^n)_+, L] = [L^n - (L_n)_-, L] = -[(L^n)_-, L].$$

Now the leftmost commutator is an operator of non-negative order. But $(L^n)_-$ starts with at least $D^{-1}$, so the rightmost commutator is of order at most $(2 - 1 - 1) = 0$. Thus $[(L^n)_+, L]$ contains no $D$’s at all, and is just a function.

We can at this point give another explanation of why the flows commute. Consider an eigenfunction of $L$

$$L\psi(t, k) = k^2 \psi(t, k).$$

Here $t$ denotes the set of all $t_n$. We have

$$\frac{\partial}{\partial t_n} \psi = Q_n \psi$$

where $Q_n = (L^{n/2})_+$ so the condition that $\psi$ (and hence $L$) be a well-defined function of all the $t$ is that

$$\frac{\partial^2 \psi}{\partial t_n \partial t_m} = \frac{\partial^2 \psi}{\partial t_m \partial t_n}.$$
or that
\[
\frac{\partial Q_m}{\partial t_n} - \frac{\partial Q_n}{\partial t_m} - [Q_n, Q_m] = 0.
\]
These are known as the Zakharov-Shabat equations. We verify that they are true:

\[
\text{LHS} = (\partial_n L^{m/2} - \partial_m L^{n/2} + [Q_m, Q_n])_+
\]
\[
= ([Q_n, L^{m/2}] - [Q_m, L^{n/2}] + [Q_m, Q_n])_+
\]
\[
= (Q^m - L^{m/2}, Q_n - L^{n/2})_+
\]
\[
= ([-(L^{m/2})_-, -(L^{n/2})_+])_+
\]
\[
= 0.
\]

**KP Hierarchy**

The KdV Hierarchy does not make use of the even \(t_n\). A more general setting is obtained by replacing the KdV operator \(L\) with the operator

\[
L = D + u_0(x)D^{-1} + u_1(x)D^{-2} + \cdots, \quad (\text{KP})
\]

and taking the time evolutions to be

\[
\partial_t L = [(L^m)_+, L].
\]

(Observe that under these evolutions the coefficient of \(D^1\) remains unity, and the coefficient of \(D^0\) remains zero.) If we set \(B_m = (L^m)_+\), then the same algebra as above gives us

\[
\partial_t B_m - \partial_m B_n - [B_m, B_n] = 0.
\]

The flows therefore commute as for KdV. Indeed, in the special case that \(L^2\) contains no negative powers of \(D\), this set of equations reduces to the KdV set.

The equations (KP) form a coupled set of infinitely-many equations in an infinite set of variables. Not a nice thing! Each individual Zakharov-Shabat compatibility equation, however, provides a finite set of equations in a finite set of variables. For example, if we take \(n = 3, m = 2\), then only \(u_0\) and \(u_1\) are involved. Write \(2u_0 = u, \ t_1 = t\) and \(t_2 = y\), then (after a fair bit of labour) we that

\[
\begin{align*}
  u_y &= u_{xx} + 4u_x, \\
  u_{yx} + 2(u_1)_y - \frac{2}{3}u_t &= \frac{1}{2}u_{xxx} + 2(u_1)_{xx} - uu_x.
\end{align*}
\]
Eliminating $u_1$ leads to

\[ 3u_{yy} = (4u_t - u_{xxx} - 6uu_x)_x \]

This is the eponymous Kadomtsev-Petviashvili (KP) equation.

**Baker function**

We used the idea that the formal (we do not care about boundary conditions here) eigenfunction

\[ L\psi = \lambda \psi \]

is a function of all of the $(t_1, t_2, t_3 \ldots)$ but have not discussed its form. For KdV with $L = D^2$ it is $e^{kx}$ and $\lambda = k^2$. As $q(x)$ becomes non-zero, we can look for solutions of the form

\[ \psi(x, t) = e^{kx} \left( 1 + \frac{w_1(x, t)}{k} + \frac{w_2(x, t)}{k^2} + \cdots \right). \]

When $q(x)$ is a many-soliton solution of the KdV equation (and for $k \to i\kappa$) the Schrödinger equation really does have solutions of this form because such $q(x)$ are reflectionless potentials. For general KdV and KP solutions, however, the series is at best asymptotic. Nonetheless the $w_n$ coefficients encode the almost the same information as the $u_n$ defining $L$, and the formal series provides a tool for working with them. I say “almost” the same, because we can multiply $\psi(x, t)$ by a series $1 + a_1/k + a_2/k^2 + \cdots$ with constant $a_n$’s without affecting the $u_n$ or $L$.

At this point it is convenient to define the action of $D^{-1}$ on exponentials by setting

\[ D^n e^{kx} = (k)^n e^{kx}, \quad \text{for all } n \in \mathbb{Z}. \]

We can then write

\[ \psi(x, t) = (1 + w_1(x, t)D^{-1} + w_2(x, t)D^{-2} + \cdots) e^{kx}. \]

Now we do several things that are for now mysterious. Firstly we change labels on the set of multi-times $t_n$ to $x_n$. We will also take $t_1 = x_1$ to be synonymous with the co-ordinate $x$. We can do this because $\partial_t u_n = \partial_x u_n$ tells us that the $u_n$ contain $x$ and $x_1$ only in the combination $x + x_1$. So we don’t need both. Lastly, it turns out to be useful to multiply $\psi$ by a factor that puts all the $x_n$ on an equal footing in the exponential factor. We therefore define

\[ w(x, k) = (1 + w_1(x, t)D^{-1} + w_2(x, t)D^{-2} + \cdots) \exp\{\xi(x, k)\}, \]
where
\[ \xi(x, k) = \sum_{n=1}^{\infty} x_n k^n. \]

The object \(w(x, k)\) is known as the Baker-Akhiezer function. The prefactor
\[ \Phi = 1 + w_1(x)D^{-1} + w_2(x)D^{-2} + w_3(x)D^{-3} \ldots \]
is the dressing operator. We can use it to write
\[ L = \Phi D \Phi^{-1}, \]
as the right- and left-hand sides have the same effect on all eigenfunctions. I claim that
\[ \frac{d\Phi}{dx} = -(L^n)_- \Phi. \]

This time evolution makes
\[
\frac{dL}{dx_n} = \partial_{x_n} (\Phi D \Phi^{-1}) = (\partial_{x_n} \Phi) D \Phi^{-1} - \Phi D \Phi^{-1} (\partial_{x_n} \Phi) \Phi^{-1} = -(L^n)_- (\Phi D \Phi^{-1}) (L^n)_-
\]
as before. It also leads to
\[
\frac{dw}{dx_n} = \left( \partial_{x_n} \Phi \right) e^{\xi(x, k)} + \Phi \left( \partial_{x_n} e^{\xi(x, k)} \right) = -(L^n)_- \Phi e^{\xi(x, k)} + \Phi k^n e^{\xi(x, k)} = -(L^n)_- \Phi e^{\xi(x, k)} + L^n \Phi e^{\xi(x, k)} = (L^n - (L^n)_-) \Phi e^{\xi(x, k)} = (L^n)_+ w(x, k),
\]
as it should.
Now let
\[ \Phi^* = 1 + (-D)^{-1} w_1(x) + (-D)^{-2} w_2(x) + (-D)^{-3} w_3(x) + \cdots \]
be the formal adjoint of $\Phi$. Then
\[
w^*(x, k) \overset{\text{def}}{=} (\Phi^*)^{-1} \exp\{-\xi(x, k)\}
\]
is called the \textit{adjoint Baker function}. The KP linear operator $L$ is not even formally self-adjoint, and we have
\[
L^* w^* = k w^*, \\
d\frac{w^*}{dx} = -B_n^* w^*.
\]

***********************

Aside 3: We can construct explicit $\Phi$ operators that generate the multisoliton solutions of KdV. For an $N$-soliton solution, we first prepare $N$ linearly independent functions
\[
y_n(x, t) = \exp\{\alpha_n x + \alpha_n^3 t\} + a_n \exp\{-\alpha_n x - \alpha_n^3 t\}.
\]
The $\alpha_n$ will parametrize the speed and amplitude of the $n$-th soliton. The $a_n$ will parametrize it’s position at $t = 0$. We then form the operator
\[
F = \frac{1}{W} \left| \begin{array}{cccc}
D_N & D_{N-1} & \cdots & D \\
y_1^{(N)} & y_1^{(N-1)} & \cdots & y_1' \\
\vdots & \vdots & \ddots & \vdots \\
y_N^{(N)} & y_N^{(N-1)} & \cdots & y_N'
\end{array} \right|,
\]
where $W(y_1, \ldots, y_N)$ is the Wronskian, and the $D^n$ are understood to be written to the right of the functions. The operator $F$ has been constructed so that the coefficient of $D^N$ is unity, and so that $Fy_n = 0$ for $n = 1, \ldots, N$. The $y_n$ have been constructed so that
\[
\dot{y}_n = \partial_x^3 y_n, \quad \partial_x^2 y_n = \alpha_n^2 y_n.
\]
We then set
\[
\Phi = F D^{-N} = 1 + w_1(x, t) D^{-1} + \cdots + w_N(x, t) D^{-N}.
\]

We can use the above properties to show that $L = \Phi D^2 \Phi^{-1}$ is a purely differential operator of the form $D^2 + q(x, t)$, and that $q(x, t)$ obeys the KdV equation
\[
\dot{q} = \frac{1}{4} u''' + \frac{3}{2} uu'.
\]
**Bilinear identity**

Given a formal Laurent series \( f(k) = \sum a_n k^n \), we define its *residue* to be

\[
\text{Res}_k \{ f(k) \} = a_{-1}.
\]

Given suitable analytic conditions on \( f \), we could write

\[
\text{Res}_k \{ f(k) \} = \oint f(k) \frac{dk}{2\pi i},
\]

for some contour encircling \( k = \infty \), but for now the formal algebraic definition will suffice. For

\[
P = \sum_{k=0}^{\infty} p_{n-k}(x) D^{n-k}
\]

we define

\[
\text{Res}_D \{ P \} = p_{-1}(x)
\]

Aside 4: A useful fact is that

\[
\text{Res}_D \{ [P, Q] \} = h'(x),
\]

for some function \( h(x) \). Thus for operators with suitable coefficient functions we have

\[
\int_{-\infty}^{\infty} \text{Res}_D \{ [P, Q] \} \ dx = 0.
\]

From this fact we can deduce that

\[
J_n[u] \overset{\text{def}}{=} \int_{-\infty}^{\infty} \text{Res}_D \{ L^n \} \ dx
\]

has time derivatives

\[
\frac{dJ_n}{dx_m} = \int_{-\infty}^{\infty} \text{Res}_D \{ B_m, L^n \} \ dx = 0,
\]

and are are conserved quantities under all the multi-time evolutions.
We use the residue to express a key, but rather mysterious, property of the Baker function and its adjoint. Let $x$ and $x'$ be two sets of multi-times, then we claim that

$$\text{Res}_k \{ w(x,k) w^*(x',k) \} = 0.$$  

This is the bilinear identity. We will show that it is entirely equivalent to the evolution equations.

We begin with a short lemma: Let $P = \sum p_n D^n$, and $Q = \sum q_n D^n$, then

$$\text{Res}_k \{(P e^{kx})(Q e^{-kx})\} = \text{Res}_D \{P Q^*\}.$$  

The proof is a simple computation:

$$\text{Res}_k \{(P e^{kx})(Q e^{-kx})\} = \text{Res}_k \left( \sum_i p_i k^i \sum_j q_j (-k)^j \right) = \sum_{i+j=-1} (-1)^j p_i q_j.$$  

Similarly

$$\text{Res}_D \{P Q^*\} = \text{Res}_D \left( \sum_i p_i D^i \sum_j (-D)^j q_j \right) = \sum_{i+j=-1} (-1)^j p_i q_j.$$  

Now we show that

$$\text{Res}_k \{(\partial_{x_1}^{n_1} \ldots \partial_{x_m}^{n_m} w(x,k)) w^*(x,k)\} = 0$$  

for any set $(n_1, \ldots, n_m)$, $n_i \geq 0$. Granted a Taylor expansion, this is equivalent to the bilinear identity.

Because $\partial_{x_m} w = B_m w$ allows us to trade off derivatives with respect to $x_m$ for those with respect to $x_1$, we only need to prove this for $(n_1,0\ldots,0)$, $n > 0$. Now

$$\text{Res}_k \{(\partial_{x_1}^n w(x,k)) w^*(x,k)\} = \text{Res}_k \{((\partial_{x_1}^n \Phi e^{\xi(x,k)})(\Phi^*)^{-1} e^{-\xi(x,k)})\}$$  

$$= \text{Res}_k \{((\partial_{x_1}^n \Phi e^{x_1 k})(\Phi^*)^{-1} e^{-x_1 k})\}$$  

$$= \text{Res}_k \{(D^n \Phi e^{x_1 k})(\Phi^*)^{-1} e^{-x_1 k})\}$$  

$$= \text{Res}_D \{(D^n \Phi)^{-1}\} \quad \text{(by lemma)}$$  

$$= \text{Res}_D \{D^n\}$$  

$$= 0.$$
Now we establish the converse. Suppose we construct
\[
w(x, k) = \left(1 + \frac{w_1(x)}{k} + \frac{w_2(x)}{k^2} + \ldots\right) e^{\xi(x,k)},
\]
\[
w^*(x, k) = \left(1 + \frac{w_1^*(x)}{k} + \frac{w_2^*(x)}{k^2} + \ldots\right) e^{-\xi(x,k)}
\]
for some sets of functions \( w_n(x, k) \) and \( w_n^*(x, k) \), and assume only that \( \text{Res}_k \{ (\partial^{n_1}_{x_1} \ldots \partial^{n_m}_{x_m} w(x, k)) w^*(x, k) \} = 0 \)
for any set \((n_1, \ldots, n_m), n_i \geq 0\). We set
\[
\Phi = 1 + w_1(x)D^{-1} + w_2(x)D^{-2} + \ldots,
\]
and
\[
\Psi = 1 + w_1^*(x)(-D)^{-1} + w_2^*(x)(-D)^{-2} + \ldots
\]
so that
\[
w(x, k) = \Phi e^{\xi(x,k)}, \quad w^*(x, k) = \Psi e^{-\xi(x,k)}.
\]
Then
\[
\text{Res}_D \{ D^n \Phi \Psi^* \} = \text{Res}_k \{ (D^n \Phi e^{\xi(x,k)}) (\Psi e^{-\xi(x,k)}) \}
\]
\[
= \text{Res}_k \{ (\partial^n_{x_n} w) w^* \}
\]
\[
= 0, \quad \text{by our assumption.}
\]
This is true for all \( n > 0 \). But \( \Phi \Psi^* = 1 + X \), where \( X \) has only negative powers of \( D \). We are saying that \( \text{Res}_D (D^n X) = 0 \) for all positive \( n \), and thus that \( X = 0 \). We deduce that \( \Psi = (\Phi^*)^{-1} \).

Now define \( L = \Phi D \Phi^{-1} \), and set \( B_m = (L^m)_+ \). We wish to show that the bilinear identity implies that \( \partial_{x_m} L = [B_m, L] \). It is sufficient to show that \( \partial_{x_n} \Phi = -(L^n)_- \Phi \). Now
\[
((\partial_{x_n} \Phi) + (L^n)_- \Phi) e^{\xi(x,k)} = (\partial_{x_n} \Phi - \Phi (L^n)_- \Phi) e^{\xi(x,k)}
\]
\[
= (\partial_{x_n} \Phi - \Phi D^n + (L^n)_- \Phi) e^{\xi(x,k)}
\]
\[
= (\partial_{x_n} \Phi - L^n \Phi + (L^n)_- \Phi) e^{\xi(x,k)}
\]
\[
= (\partial_{x_n} - L^n + (L^n)_-) \Phi e^{\xi(x,k)}
\]
\[
= (\partial_{x_n} - (L^n)_+) \Phi e^{\xi(x,k)}.
\]
In the last line \((\partial_{x_n} - (L^n)_+)\) contains positive powers of derivatives only, so, from the bilinear identity we know that

\[
0 = \text{Res}_k \left\{ \left( \partial_{x_1} (\partial_{x_n} - (L^n)_+) \Phi e^{\xi(x,k)} \right) \left( \Psi e^{-\xi(x,k)} \right) \right\} \\
= \text{Res}_k \left\{ \left( \partial_{x_1} ((\partial_{x_n} \Phi) + (L^n)_- \Phi) e^{\xi(x,k)} \right) \left( \Psi e^{-\xi(x,k)} \right) \right\} \\
= \text{Res}_D \left\{ D^n ((\partial_{x_n} \Phi) + (L^n)_- \Phi) \Psi^* \right\} \\
= \text{Res}_D \left\{ D^n ((\partial_{x_n} \Phi) + (L^n)_- \Phi) \Phi^{-1} \right\}.
\]

The last line tells us that

\[
0 = ((\partial_{x_n} \Phi) + (L^n)_- \Phi) \Phi^{-1},
\]

and hence

\[
\partial_{x_n} \Phi = -(L^n)_- \Phi,
\]

as was to be shown.

Finding solutions to the KP and other equations is therefore equivalent to finding Baker functions \(w(x,k)\) that satisfy the bilinear identity. This property is intimately connected to the existence of a “\(\tau\)-function.”

**The \(\tau\)-function**

The pattern of manipulations above seems rather pulled out of the air. We here consider a simpler system where we can see all the same things happening.

Let \(\mu(x)\) be a non negative function of compact support and define an inner product on real polynomials by

\[
\langle f, g \rangle_{\mu} = \int f(x)g(x)\mu(x) \, dx.
\]

Consider the associated monic orthogonal polynomials \(p_i(x)\) whose inner products are

\[
\langle p_i, p_j \rangle_{\mu} \equiv \int p_i(x)p_j(x)w(x) \, dx = h_i \delta_{ij}.
\]

As do all families of orthogonal polynomials, they obey a recursion relation

\[
xp_i(x) = p_{i+1}(x) + a_ip_i(x) + b_{i-1}^2 p_{i-1}(x); \quad p_{-1}(x) = 0, \quad p_0(x) = 1.
\]

Write the recursion relation as

\[
Lp = xp.
\]
where

\[
L \equiv \begin{bmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & 1 & a_2 & b_1^2 & 0 \\
\ddots & 0 & 1 & a_1 & b_0^2 \\
\ddots & 0 & 0 & 1 & a_0 \\
\end{bmatrix}, \quad p \equiv \begin{bmatrix}
\vdots \\
p_2 \\
p_1 \\
p_0 \\
\end{bmatrix}.
\]

Now suppose that

\[
\mu(x) = \exp \left\{ \sum_{n=1}^{\infty} t_n x^n \right\} \mu_0(x).
\]

We investigate how the polynomials \( p_i(x) \) and the coefficients \( a_i \) and \( b_i^2 \) vary with the parameters \( t_n \). It should be clear that

\[
\frac{\partial p}{\partial t_n} = -M^{(n)} p,
\]

where \( M^{(n)} \) is some strictly upper triangular matrix - i.e. all entries on and below its principal diagonal are zero. I’ve inserted the minus sign, because the idea is to compare the equations we will get with those for the dressing operator \( \Phi \).

By differentiating \( Lp = xp \) with respect to \( t_n \) we see that

\[
\frac{\partial L}{\partial t_n} = [-M^{(n)}, L].
\]

We now compute the matrix elements

\[
\langle i|M^{(n)}|j \rangle \equiv M^{(n)}_{ij} = h_j^{-1} \left\langle p_j, \frac{\partial p_i}{\partial t_n} \right\rangle_w
\]

(note the interchange of the order of \( i \) and \( j \) in the \( \langle , \rangle_w \) product!) by differentiating the orthogonality condition \( \langle p_i, p_j \rangle_w = h_i \delta_{ij} \). We read off that

\[
M^{(n)} = (L^n)_-
\]

where \((L^n)_-\) denotes the strictly upper triangular projection of the \( n \)'th power of \( L \) — i.e. the matrix \( L^n \), but with its diagonal and lower triangular entries replaced by zero.

Thus

\[
\frac{\partial L}{\partial t_n} = [- (L^n)_-, L]
\]
describes a family of deformations of the semi-infinite matrix \( L \) that, in some formal sense, preserve its eigenvalues \( x \).

We now define the "tau-function" \( \tau_n(t_1, t_2, t_3 \ldots) \) of the parameters \( t_i \) to be the \( n \)-fold integral

\[
\tau_n(t_1, t_2, \ldots) = \int \cdots \int \mu_0(x_1)dx_1 \cdots \mu_0(x_n)dx_n \Delta^2(x) \exp \left\{ \sum_{\nu=1}^{n} \sum_{m=1}^{\infty} t_m x_{\nu}^m \right\},
\]

where

\[
\Delta(x) = \begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{vmatrix} = \prod_{\nu<\mu} (x_\nu - x_\mu)
\]

is the \( n \)-by-\( n \) Vandermonde determinant.

From the identity

\[
\begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{vmatrix} = \begin{vmatrix} p_{n-1}(x_1) & p_{n-2}(x_1) & \cdots & p_1(x_1) & p_0(x_1) \\ p_{n-1}(x_2) & p_{n-2}(x_2) & \cdots & p_1(x_2) & p_0(x_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{n-1}(x_n) & p_{n-2}(x_n) & \cdots & p_1(x_n) & p_0(x_n) \end{vmatrix}
\]

combined with with the orthogonality property of the \( p_n(x) \) we see that

\[
p_n(x) = \frac{1}{\tau_n} \int \mu_0(x_1)dx_1 \cdots \mu_0(x_n)dx_n \Delta^2(x) \prod_{\mu=1}^{n} (x - x_\mu) \exp \left\{ \sum_{\nu=1}^{n} \sum_{m=1}^{\infty} t_m x_{\nu}^m \right\}
\]

\[
= x^n \tau_n \left( t_1 - \frac{1}{x}, t_2 - \frac{1}{2x^2}, t_3 - \frac{1}{3x^3}, \ldots \right) / \tau_n(t_1, t_2, t_3, \ldots).
\]

At the last step we have used

\[
\left( 1 - \frac{x_i}{x} \right) = \exp \left\{ \ln \left( 1 - \frac{x_i}{x} \right) \right\} = \exp \left\{ -\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{x_i}{x} \right)^n \right\}.
\]

**N-Soliton example**

A similar \( \tau \)-function formula generates solutions for real soliton equations. By this I mean that there is a \( \tau_n(x_1, x_2, x_3, \ldots) \) such that

\[
w(x, k) = \tau \left( x_1 - \frac{1}{k}, x_2 - \frac{1}{2k^2}, x_3 - \frac{1}{3k^3}, \ldots \right) e^{\xi(x,k)}/\tau_n(x_1, x_2, x_3, \ldots).
\]
Let's see this in action.

The Baker function for the N-soliton KP equation is constructed in the same way as for the KdV. We set

$$\xi(x, \alpha) = x_1 \alpha + x_2 \alpha^2 + x_3 \alpha^3 + \cdots$$

and take

$$y_n = \exp \xi(x, \alpha_n) + a_n \exp \xi(x, \beta_n)$$

so that

$$\partial_{x_m} y_n = \partial_{x_1}^m y_n.$$  

(For KdV we would take $\beta_n = -\alpha_n$.) Then set

$$F = \frac{1}{W} \begin{vmatrix} D^N & D^{N-1} & \cdots & D & 1 \\ y^{(N)}_1 & y^{(N-1)}_1 & \cdots & y'_1 & y_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y^{(N)}_N & y^{(N-1)}_N & \cdots & y'_N & y_N \end{vmatrix},$$

where, as for KdV, $W(y_1, \ldots, y_N)$ denotes the Wronskian, and the $D^n$ are understood to be written to the right of the functions. The dressing operator is then $\Phi = FD^{-N}$, and the Baker function is

$$w(x, k) = \Phi \exp \xi(x, k) = \frac{1}{W} \begin{vmatrix} 1 & k^{-1} & \cdots & k^{-N+1} & k^{-N} \\ y^{(N)}_1 & y^{(N-1)}_1 & \cdots & y'_1 & y_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y^{(N)}_N & y^{(N-1)}_N & \cdots & y'_N & y_N \end{vmatrix} \exp \xi(x, k).$$

I claim that the $\tau$ function from which this dressing function is derived is just the Wronskian

$$\tau(x_1, x_2, \ldots) = W(y_1, y_2, \ldots) = \begin{vmatrix} y^{(N-1)}_1 & \cdots & y'_1 & y_1 \\ \vdots & \ddots & \vdots & \vdots \\ y^{(N-1)}_N & \cdots & y'_N & y_N \end{vmatrix}.$$

Let us verify that this works. We need to evaluate

$$\tau \left( x_1 - \frac{1}{k}, x_2 - \frac{1}{2k^2}, \ldots \right)$$
With this shift
\[ y_n \rightarrow \exp \left\{ \xi(x, \alpha_n) - \left( \frac{\alpha_n}{k} + \frac{\alpha_n^2}{2k^2} + \cdots \right) \right\} + a_n \exp \left\{ \xi(x, \beta_n) - \left( \frac{\beta_n}{k} + \frac{\beta_n^2}{2k^2} + \cdots \right) \right\}, \]
\[ = \left( 1 - \frac{\alpha_n}{k} \right) \exp \xi(x, \alpha_n) + a_n \left( 1 - \frac{\beta_n}{k} \right) \exp \xi(x, \beta_n), \]
\[ = y_n - \frac{1}{k} y_n' \]  

(1)

So
\[ \tau \left( x_1, x_2, \ldots \right) = \frac{1}{W} \left| \begin{array}{cccc} y_1^{(N-1)} - \frac{1}{k} y_1^{(N)} & \cdots & y_1' - \frac{1}{k} y_1'' & y_1 - \frac{1}{k} y_1' \\ \vdots & \ddots & \vdots & \vdots \\ y_N^{(N-1)} - \frac{1}{k} y_N^{(N)} & \cdots & y_N' - \frac{1}{k} y_N'' & y_N - \frac{1}{k} y_N' \end{array} \right|. \]

The determinant in the expression for \( w(x, k) \) can be reduced to this by subtracting \( 1/k \) times the penultimate column from the last column, \( 1/k \) times the third from the right from the penultimate column, and so on. Thus the formula works.

We can make this \( \tau \) function look prettier, by pulling out some common factors. For \( N = 2 \) we can rearrange the Wronskian \( \tau \) function as
\[ \tau(x_1, \ldots) = C \left( 1 + A_1 e^{\xi_1} + A_2 e^{\xi_2} + A_1 A_2 \frac{(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)}{(\alpha_1 - \beta_2)(\beta_1 - \alpha_2)} e^{\xi_1 + \xi_2} \right), \]
where
\[ \xi_i = \xi(x, \beta_i) - \xi(x, \alpha_i), \]
\[ etc., \]
\[ C = (\alpha_2 - \alpha_1) e^{\xi(x, \alpha_1) + \xi(x, \alpha_2)}, \]
and
\[ A_1 = a_1 \frac{(\alpha_2 - \beta_1)}{\alpha_2 - \alpha_1}, \quad A_2 = a_2 \frac{(\alpha_2 - \alpha_1)}{\alpha_2 - \alpha_1}. \]

Now the pulled-out factor \( C \) only multiplies \( w(x, k) \) by an \( x \)-independent factor. It does not change \( L \), and can be discarded.

For the \( N \) soliton case, we similarly obtain
\[ \tau(x_1, x_2 \ldots) = C \sum_{J \subseteq I} \left( \prod_{i \in I} A_i \right) \left( \prod_{i, j \in J; i < j} c_{ij} \right) \exp \left( \sum_{i \in J} \xi_i \right). \]
where \( C \) is again disposable, and
\[
c_{ij} = \frac{(\alpha_i - \alpha_j)(\beta_i - \beta_j)}{(\beta_i - \alpha_j)(\alpha_i - \beta_j)}.
\]
For \( N = 3 \), for example
\[
\tau(x_1, x_2, \ldots) = C \left( 1 + A_1 e^{\xi_1} + A_2 e^{\xi_2} + A_3 e^{\xi_3} + A_1 A_2 c_{12} e^{\xi_1+\xi_2} + A_1 A_3 c_{13} e^{\xi_1+\xi_3} + A_2 A_3 c_{23} e^{\xi_2+\xi_3} + A_1 A_2 A_3 c_{12} c_{13} c_{23} e^{\xi_1+\xi_2+\xi_3} \right).
\]

There is clearly something going on here. The output is far simpler than the complicated-looking Wronskian would lead us to expect. What Sato and his colleagues realized is that these highly-structured expressions can be generated by the use of objects that originated in the early days of String Theory. Indeed they date from before strings were thought of and what is today “String Theory” was known as the “Dual Resonance model.” In this model, each scattering amplitude was given by contour integral of a rational function — a generalized version of the Euler Beta integral — and the rational function was generated by a collection of “Vertex operators” — one operator for each external particle entering the scattering process.

A vertex operator is an expression such as
\[
X(\beta, \alpha) \overset{\text{def}}{=} \exp \left\{ \sum_{n=1}^{\infty} (\beta^n - \alpha^n) x_n \right\} \exp \left\{ - \sum_{n=1}^{\infty} \frac{1}{n} (\beta^{-n} - \alpha^{-n}) \frac{\partial}{\partial x_n} \right\}.
\]

Now, if \( A, B \) is a \( c \)-number we have
\[
e^{A}e^{B} = e^{B}e^{A}[A,B],
\]
and so
\[
X(\beta_1, \alpha_2)X(\beta_1, \alpha_2) = \frac{(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)}{(\alpha_1 - \beta_2)(\beta_1 - \alpha_2)} : X(\beta_1, \alpha_2)X(\beta_1, \alpha_2) :,
\]
where the normal-ordering symbol : : means that all the derivative terms have been passed to the right. Then, for example
\[
\exp\{a_1 X(\beta_1, \alpha_1)\} \exp\{a_2 X(\beta_2, \alpha_2)\} \cdot 1 = (1 + a_1 X(\beta_1, \alpha_1))(1 + a_2 X(\beta_2, \alpha_2)) \cdot 1
\]
\[
= 1 + a_1 e^{\xi_1} + a_2 e^{\xi_2} + a_1 a_2 \frac{(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)}{(\alpha_1 - \beta_2)(\beta_1 - \alpha_2)} e^{\xi_1+\xi_2}.
\]

Who ordered these stringy objects?

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Aside 5: The quantities \[ \frac{\partial^2 \ln \tau}{\partial x_n \partial x_m} \]

turn out to be differential polynomials in the \( u_n(x) \). In particular

\[ j_n = \frac{\partial^2 \ln \tau}{\partial x_1 \partial x_n} = \text{Res}_D \{ L^n \}. \]

These facts are not obvious, and they imply the conservation laws: by interchanging the order of differentiation, we have

\[ \frac{\partial j_n}{\partial x_i} = \frac{\partial}{\partial x_1} \left( \frac{\partial^2 \ln \tau}{\partial x_i \partial x_n} \right) \]

which shows that the integrals of the \( J_n = \int j_n \, dx \) are conserved by all the time evolutions (a fact we noted earlier). In particular, for KdV we have

\[ j_1 = \frac{\partial^2 \ln \tau}{\partial x_1 \partial x_1} = q(x)/2. \]

Thus the \( \tau \)-function expression we wrote down above provides an explicit formula for the \( N \)-soliton solution.

*********

Bosonization and Vertex Operators

Consider right-moving relativistic fermions moving on a circle of circumference \( 2\pi \). Take \( H = -i \partial \theta + 1/2 \) to be the Hamiltonian. A basis of single-particle eigenstates is then the set of functions \( e^{i n \theta} = z^n \). The ground state is a Dirac sea in which the states with \( n < 0 \) are occupied and those with \( n \geq 0 \) empty. Now an infinitely deep Dirac sea can be treacherous. It is safer to begin with a finite number of particles \( 1, \ldots, N \), filling a shallow sea. For the moment assume that the sea bed corresponds to the state \( z^0 = 1 \), and its surface to \( z^{N-1} \). Later we can take \( N \to \infty \) and renormalize the origin of the momentum quantum number so that the Fermi surface is \( n = 0 \).

The Hilbert space of \( N \)-body wavefunctions is now spanned by occupation-number eigenstates which are \( N \times N \) Slater determinants of the \( z^n, n \geq 0 \). The ground state is the Vandermonde determinant

\[ \Psi_0(z_1, z_2, \ldots, z_N) = \begin{vmatrix} z_1^{N-1} & z_2^{N-2} & \cdots & 1 \\ z_2^{N-1} & z_1^{N-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ z_N^{N-1} & z_N^{N-2} & \cdots & 1 \end{vmatrix} = \prod_{i<j} (z_i - z_j). \]
We will frequently denote this determinant by the symbol $\Delta(z)$.

A general Slater determinant of the single-particle states can be written

$$\Psi_{\{\lambda\}}(z_1, z_2, \ldots, z_N) = \left| \begin{array}{c}
  z_{1}^{\lambda_1 + N - 1} \quad z_{2}^{\lambda_2 + N - 2} \quad \cdots \quad z_{N}^{\lambda_N} \\
  z_{1}^{\lambda_1 + N - 1} \quad z_{2}^{\lambda_2 + N - 2} \quad \cdots \quad z_{2}^{\lambda_2} \\
  \vdots \quad \vdots \quad \ddots \quad \vdots \\
  z_{N}^{\lambda_1 + N - 1} \quad z_{2}^{\lambda_2 + N - 2} \quad \cdots \quad z_{N}^{\lambda_N}
\end{array} \right|$$

A compact notation is

$$\Psi_{\{\lambda\}}(z) = \det |z_{s}^{\lambda_i + N - t}|.$$ 

Because of the identity of the particles we can, and will, arrange the columns of the determinant so that $\lambda_1 \geq \lambda_2 \geq \lambda_3$ etc. $\Psi_{\{\lambda\}}$ represents a charge-neutral excitation of the ground state. If we expand out the determinant, each term has $|\{\lambda\}| \equiv \lambda_1 + \lambda_2 + \ldots + \lambda_N$ extra powers of $z$. Thus $\Psi_{\{\lambda\}}$ is an energy eigenstate with energy $E = |\{\lambda\}|$ above the ground state. I like to think of $\Psi_{\{\lambda\}}$ being created by sliding the $i$-th electron up through $\lambda_i$ steps in energy.

The labels $\{\lambda\}$ are traditional displayed as Young diagrams with $\lambda_i$ boxes in the $i$'th row. For example

```
+---+---+---+---+
|   |   |   | |
|   |   |   | |
|   |   |   | |
|   |   |   | |
```

represents $\{4332\}$, or as it sometimes convenient to write $\{43^22\}$.

This pictorial representation of the neutral excitations has nice properties under particle-hole interchange: If we were, for example, to take the topmost electron and move it up four steps, we would represent the resulting state label by the Young diagram $\{4\}$:

```
+---+---+---+---+
|   |   |   |   |
|   |   |   |   |
|   |   |   |   |
```

The particle-hole conjugate of this state has a hole four steps down in the sea. This state is made by raising the top four electrons each by one step, and its label is represented by $\{1111\}$, or equivalently $\{1^4\}$:

```
```

i.e as the conjugate diagram obtained by interchanging the rows and columns of the Young diagram. This is true in general — interchanging particles with holes interchanges rows and columns.
The set of distinct $\lambda_i$ with $\sum \lambda_i = M$ are in one-to-one correspondence with the partitions of $M$. It is easy to see that generating function of the number, $p(M)$, of distinct partitions of $M$ is given by

$$Z = \prod_{n>0} \frac{1}{(1 - x^n)} = \sum_M p(M)x^M.$$  

The generating function $Z(x)$ is known in number theory as the “partition function”. Now if we put $x = e^{-\beta}$ we obtain the statistical-mechanics partition function for the charge neutral excitations,

$$Z = \sum_M p(M)e^{-\beta M} = \frac{1}{\prod_{n>0} (1 - e^{-\beta n})}.$$

The right hand side is recognizable as the partition function of an infinite ensemble $n = 1, 2, \ldots, \infty$ of bosonic oscillators of frequency $n$, and so already hints at bosonic description of the states.

To identify the bosons, we make a connection with the theory of symmetric functions. We first observe that the Vandermonde determinant, $\Delta(z) \equiv \Psi_0 \equiv \Psi_{\{\emptyset\}}$, is a factor of all the $\Psi_{\{\lambda\}}(z)$. This means that the quotient

$$\Phi_{\{\lambda\}}(z) = \frac{\Psi_{\{\lambda\}}(z)}{\Psi_0(z)}$$

is a symmetric polynomial in the $z_i$. Clearly all charge-neutral states $\Psi_{\{\lambda\}}$ are obtained by multiplying $\Psi_0$ by one of these symmetric polynomials, so the Hilbert space of charge-neutral excitations is isomorphic to the linear space spanned by them.

The polynomial $\Phi_{\{\lambda\}}(z)$ is the Schur function associated with the partition $\{\lambda\}$ of $M$. These functions are best known in physics as the characters of the groups $GL(n)$, $U(n)$, or $SU(n)$, where $n$ is the number of rows in the Young diagram. The famous diagrammatic recipe — known to mathematicians as the Littlewood-Richardson rule — for combining Young diagrams so as to find the representations occurring in the Clebsh-Gordan decomposition of direct products is nothing but the algorithm for decomposing products of the $\Phi_{\{\lambda\}}(z)$ into sums of $\Phi_{\{\lambda'\}}(z)$’s. Our interest, however, lies principally in the role they play in the theory of symmetric functions.

Symmetric functions.

Given a set of symbols $z_i$, we form the ring $S(z)$ of polynomials in the $z_i$ which are invariant under arbitrary permutations of the $z_i$. Elements of this
ring include the *elementary symmetric functions* $e_n$, $n = 0, \ldots, N$, that are
declared by
\[ \prod (1 - z_t) = e_0 - e_1 t + e_2 t^2 + \cdots \pm e_N. \]
A second collection comprises the *complete homogeneous symmetric polynomials* $h_i$, $i = 0, \ldots, \infty$, defined by
\[ \frac{1}{\prod (1 - z_t)} = h_0 + h_1 t + h_2 t^2 + h_3 t^3 + \cdots \]
For example, for three $z_i$ we have
\[ e_2 = z_1 z_2 + z_2 z_3 + z_3 z_1, \]
and
\[ h_3 = z_1^3 + z_2^3 + z_3^3 + z_1^2 z_2 + z_1^2 z_3 + z_2^2 z_3 + z_2^2 z_1 + z_3^2 z_1 + z_3^2 z_2 + z_1 z_2 z_3. \]
By inverting the power series definitions, we can express the $e_n$ as polynomials in the $h_n$ and *vice versa*. Remarkably the coefficients are integers in both directions. It is a classical theorem that sums of products of either the $e_n$ or the $h_n$ generate the ring $S(z)$.

The *power sums* $s_n$, $n > 0$, are defined as
\[ s_n(z) = \sum_i z_i^n. \]
By using the relation
\[
\exp \left\{ \sum_{n=1}^{\infty} \frac{1}{n} t^n s^n(z) \right\} = \exp \left\{ - \sum_i \ln(1 - z_t) \right\} = \prod_i \exp \{- \ln(1 - z_t)\} = \prod_i \frac{1}{1 - z_t} = 1 + h_1 t + h_2 t^2 + h_3 t^3 + \cdots
\]
we can express the $h_n$ as polynomials in the $s_n$ (with rational coefficients), and, by reverting the series, express the $s_n$ as polynomials in the $h_n$ (with
integer coefficients). Consequently sums of products of the \(s_n\) also generate \(S(z)\).

While it is not obvious, the Schur functions can be written (the Jacobi-Trudi formula) in terms of the \(h_n\) as

\[
\Phi_{\{\lambda\}}(\alpha) = \det |h_{\lambda_i-i+j}|.
\]

For example

\[
\Phi_{\{p\}} = h_p,
\]

and

\[
\Phi_{\{p,q,r\}} = \begin{vmatrix}
  h_p & h_{p+1} & h_{p+2} \\
  h_{q-1} & h_q & h_{q+1} \\
  h_{r-2} & h_{r-1} & h_r
\end{vmatrix}.
\]

Notice the pattern. The sequence \(\lambda_i\) provides the suffices on the entries on the diagonal. The suffices increase to the right and decrease to the left. We set \(h_0 = 1\) and \(h_n = 0\) if \(n < 0\).

We can also easily evaluate

\[
\Phi_{\{1^N\}}(z) = z_1 z_2 \ldots z_N = e_N.
\]

This is an example of the expression for \(\Phi_{\{\lambda\}}\) in terms of the \(e_n\). In general replacing the \(h_n\)'s by the \(e_n\)'s and replacing the partition by its conjugate gives the same Schur function. For example

\[
\Phi_{\{322\}} = \begin{vmatrix}
  h_3 & h_4 & h_5 \\
  h_1 & h_2 & h_3 \\
  1 & h_1 & h_2
\end{vmatrix} = \begin{vmatrix}
  e_3 & e_4 & e_5 \\
  e_2 & e_3 & e_4 \\
  0 & 1 & e_1
\end{vmatrix}.
\]

In the first expression the diagonal suffixes \(\{3, 2, 2\}\) come from the diagram

\[
\begin{array}{c}
  \square \\
  \square \\
  \square \\
\end{array}
\]

and in the second from the conjugate partition \(\{3, 3, 1\}\)

\[
\begin{array}{c}
  \square \\
  \square \\
  \square \\
\end{array}
\]

None of the relations between the \(h_n\), \(a_n\), \(s_n\), and \(\Phi_{\{\lambda\}}\) depend explicitly on the number \(N\) of the \(z_i\)’s. This is a great advantage because they provide
formulæ that do not grow in complexity as \( N \) becomes large. There is a problem when \( N \) is finite however: the \( e_i \) for \( i > N \) are zero, as are the Schur functions with more than \( N \) rows in the partition \( \{ \lambda \} \). Similarly only the first \( N \) of the \( h_n \) and \( s_n \)'s are functionally independent. To overcome these constraints we imagine that \( N \) is larger than any \( n \) we might be interested in. In this case the polynomials become functionally independent, and we speak of the ring of universal symmetric polynomials.

The Schur functions form a linear basis for the ring \( S(z) \). Any element of \( S(z) \) is a linear combination of \( \Phi_{\{\lambda\}} \)'s. To prove this observe that any symmetric polynomial in the \( z \)'s can be converted to an antisymmetric polynomial by multiplying by the Vandermonde determinant \( \Delta(z) \). Then antisymmetrising each individual monomial in the resulting expression converts the monomial to a \( \Psi_{\{\lambda\}}(z) \). Finally dividing out the catalytic \( \Delta(\alpha) \) returns the symmetric polynomial as a sum of \( \Phi_{\{\lambda\}} \).

There is an important relation connecting the \( \Phi \)'s and the \( s_k \)'s: Take two sets of symbols, \( z_i \) and \( \zeta_j \), then

\[
\exp \sum_k \frac{1}{k} s_k(z) s_k(\zeta) = \sum_{\{\lambda\}} \Phi_{\{\lambda\}}(z) \Phi_{\{\lambda\}}(\zeta). 
\]

"very useful formula"

The sum on the right hand side is over all partitions of all positive integers.

To drive this formula we start from

\[
\sum_1^\infty \frac{1}{n} s_n(z) s_n(\zeta) = \sum_1^\infty \frac{1}{n} \sum_i z_i^n \sum_j \zeta_j^n \\
= - \sum_{i,j} \ln(1 - z_i \zeta_j) \\
= - \ln \prod_{i,j} (1 - z_i \zeta_j). 
\]

Thus the left hand side is

\[
\frac{1}{\prod_{i,j} (1 - z_i \zeta_j)}. 
\]

Next we use Cauchy’s determinant identity in the form

\[
\det \left| \frac{1}{1 - z_i \zeta_j} \right| = \frac{\Delta(z) \Delta(\zeta)}{\prod_{ij} (1 - z_i \zeta_j)}, 
\]
to get
\[ \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{n} s_n(z)s_n(\zeta) \right\} = \frac{1}{\Delta(z)\Delta(\zeta)} \det \left| \frac{1}{1 - z_i\zeta_j} \right|. \]

Finally we expand out the each of the entries in the determinant on the right hand side in a geometric series and rearrange the monomial terms into \( \Psi_{(\lambda)}(z) \) and \( \Psi_{(\lambda)}(\zeta) \)’s. Dividing by the \( \Delta \)’s converts the \( \Psi_{(\lambda)} \)’s into \( \Phi_{(\lambda)} \)’s and so completes the proof.

**********

Aside 6: A quick review of boson coherent states. Let \( a, a^\dagger \) be boson annihilation and creation operators obeying \([a, a^\dagger] = 1\) and let \( |0\rangle \) be the no-particle state obeying \( a|0\rangle = 0 \). Then the \( n \)-boson normalized state is
\[ |n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle. \]

Define the un-normalized coherent states \(|z\rangle\) and \(<z|\) by
\[ |z\rangle = \exp\{\bar{z}a^\dagger\}|0\rangle, \quad <z| = <0| \exp\{za\}. \]

Note that \(|z\rangle\) contains only \( \bar{z} \) and is anti-analytic, while \(<z|\) contains only \( z \) and so is analytic. The overlap of these states is.
\[ <z_1|z_2> = \exp\{z_1\bar{z}_2\} \]

Then the coherent-state wavefunction corresponding to \(|\psi\rangle\) is the analytic function.
\[ \psi(z) = <z|\psi>. \]

For example
\[ <z|n\rangle = \frac{1}{\sqrt{n!}} z^n. \]

Now if \( d^2z \) denotes \( dx \, dy \), we have the easy integral
\[ \int_C \frac{d^2z}{\pi} z^n z^m \exp\{-|z|^2\} = \delta_{nm} n! \]

This leads to the over-completeness realtion
\[ \int_C \frac{d^2z}{\pi} |z\rangle <z| \exp\{-|z|^2\} = \sum_m |m\rangle <m| = \mathbb{I}. \]
The usual inner product of two states is therefore

\[ \langle \psi | \chi \rangle = \int_C \frac{d^2 z}{\pi} \langle \psi | z \rangle \langle z | \chi \rangle \exp\{-|z|^2\} = \int_C \frac{d^2 z}{\pi} \psi(z) \chi(z) \exp\{-|z|^2\}. \]

We also have

\[ \langle z | a^\dagger | \psi \rangle = z \langle z | \psi \rangle, \quad \langle z | a | \psi \rangle = \frac{\partial}{\partial z} \langle z | \psi \rangle. \]

The resulting Hilbert space of analytic functions, with the maps

\[ a \mapsto \frac{\partial}{\partial z}, \quad a^\dagger \mapsto z, \]

is called Bargmann-Fock space.

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The algebra \( \mathcal{S}(z) \) is both a vector space spanned by the \( \Phi_{\{\lambda\}} \) and a ring generated by the \( s_n(z) \). We now define an inner product on \( \mathcal{S}(z) \) which serves to make it into a Hilbert Space. For this purpose we wish the functions \( \Phi_{\{\lambda\}} \) to form an orthonormal basis. We could achieve this by taking the number \( N \) of the \( z_i \) to be infinite. It is technically simpler to proceed as described in the last section and merely take \( N \) sufficiently large that all the states we actually use are independent. This avoids all problems of convergence of infinite determinants. It is also a physically reasonable approach since, at least in condensed matter applications, the Fermi/Dirac sea always has finite depth. What we need in practice is that the energy scale of relevant physical processes be much smaller than the bandwidth.

The natural inner product on \( \mathcal{S}(z) \) is the one it inherits from the \( N \)-body Hilbert space spanned by the fermion occupation number eigenstates \( \Psi_{\{\lambda\}}(z) \):

\[ \langle \Phi_{\{\lambda\}} | \Phi_{\{\lambda'\}} \rangle_F = \frac{1}{N!} \int d\theta_1 \ldots d\theta_N |\Delta(z)|^2 \Phi_{\{\lambda\}}(e^{i\theta}) \Phi_{\{\lambda'\}}(e^{i\theta}). \]

The measure factor \( |\Delta(z)|^2 \) cancels the denominator of the Schur functions so

\[ \langle \Phi_{\{\lambda\}} | \Phi_{\{\lambda'\}} \rangle_F = \langle \Psi_{\{\lambda\}} | \Psi_{\{\lambda'\}} \rangle = \delta_{\{\lambda\}\{\lambda'\}}. \]
We can define an apparently different inner product on $S(z)$ by defining it on polynomials in the $s_n(z)$. Regard the $s_n$ as being independent complex variables and set

$$\langle f(s)|g(s) \rangle_B = \int_{CN} \prod_k \left[ \frac{d^2 s_k}{\pi k} \right] \overline{f(s)} g(s) e^{-\sum_{k=1}^{N} \frac{1}{\pi} |s_k|^2},$$

where $N$ is again arbitrarily large. The product of the $s_n$’s themselves is

$$\langle s_n|s_{n'} \rangle_B = n\delta_{n,n'}.$$

A general element in the ring is a sum of products of the $s_n$ such as

$$s^{(l)}(z) \equiv s_{l_1}^{l_1} s_{l_2}^{l_2} \cdots s_{l_n}^{l_n}$$

and the inner product of two such monomials is

$$\langle s^{(l)}|s^{(l')} \rangle_B = (l_1! l_2! \cdots l_n!) \delta_{(l)(l')}.$$

This second inner product is essentially that of a bosonic Bargmann-Fock (or coherent state) space where a commuting family of creation operators $a_n^\dagger$ is represented by multiplication by $s_n$. The corresponding annihilation operators, $\hat{a}_n$, are their hermitian adjoints with respect to this product. These are the derivations

$$s_n^\dagger = n \frac{\partial}{\partial s_n}.$$

(Do not confuse these operators with the complex conjugate variable $\bar{s}_n$ appearing in the inner-product integral.) Our new bosonic Hilbert space is therefore isomorphic to the space created from a cyclic vector $|0\rangle$ by application of bosonic $\hat{a}_n^\dagger$’s whose commutation relations are

$$[a_n, a_{n'}^\dagger] = n \delta_{n,n'} \quad n > 0.$$

If we define $a_{-n} = a_n^\dagger$ we can write

$$[a_n, a_{n'}] = n \delta_{n+n',0},$$

These are the commutation relations of the affine Lie algebra $\hat{u}_1$.

The remarkable fact which lies behind bosonization is that these two inner products, $\langle .| . \rangle_F$, and $\langle .| . \rangle_B$, coincide. To see, we this use the Bargman-Fock reproducing-kernel identity

$$\int \left[ \prod_k \frac{d^2 s_k(z)}{\pi k} \right] F(s_k) e^{-\sum_k \frac{1}{\pi} |s_k|^2} e^{\sum_k \frac{i}{\pi} s_k(z) t_k} = F(t_k)$$

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and the identity from the last section

\[ \exp \left\{ \sum_k \frac{1}{k} \, s_k t_k \right\} = \sum_{\{\lambda\}} \Phi_{\{\lambda\}}(s_k) \Phi_{\{\lambda\}}(t_k) \]

with the choice \( F(s_k) = \Phi_{\{\mu\}}(s_k) \). Here the Schur functions are to be expressed as a polynomial in the \( s_k(z) \)'s and \( t_k \equiv s_k(\zeta) \)'s.

Since the \( \Phi_{\{\lambda\}} \) form an independent set, comparing coefficients gives

\[ \langle \Phi_{\{\lambda\}} | \Phi_{\{\mu\}} \rangle_B = \int \prod \frac{d^2 s_k}{\pi k} \bar{\Phi}_{\{\lambda\}} \Phi_{\{\mu\}} e^{-\sum_k \frac{1}{k} |s_k|^2} = \delta_{\{\lambda\}\{\mu\}}. \]

So the \( \Phi_{\{\lambda\}} \) which form an orthonormal set with respect to \( \langle , \rangle_F \) also do so with respect to \( \langle , \rangle_B \). Thus \( \langle a, b \rangle_F = \langle a, b \rangle_B \) for all \( a, b \in S \).

An alternative demonstration of the equivalence of the two inner products is via Frobenius’ formula connecting the characters of the permutation group \( S_n \) with the characters of \( \text{GL}(n) \). Frobenius’ formula asserts that

\[ s^{(l)}(z) = \sum_{\{\lambda\}} \chi^{(\lambda)}_{(l)} \Phi_{\{\lambda\}}(z) \]

where the \( \chi^{(\lambda)}_{(l)} \) are the characters of the representation \( \{\lambda\} \) of the permutation group on \( n = l_1 + 2l_2 + 3l_3 + \ldots = |\{\lambda\}| \) symbols. The conjugacy classes of the group are labeled by \( (l) \). As group characters the \( \chi^{(\lambda)}_{(l)} \) obey the orthogonality conditions

\[ \frac{1}{|G|} \sum_{(l)} g(l) \chi^{(\lambda)}_{(l)} \chi^{(\lambda')}_{(l)} = \delta_{\{\lambda\}\{\lambda'\}} \]

where \(|G| = n!\) is the order of the permutation group and

\[ g(l) = \frac{n!}{1! l_1! 2! l_2! 3! l_3! \ldots} \]

is the number of permutation group elements in the conjugacy class \( (l) \). The Frobenius formula can be inverted to give \( \Phi_{\{\lambda\}} \) in terms of the \( S_k \):

\[ \Phi_{\{\lambda\}}(z) = \frac{1}{|G|} \sum_{(l)} g(l) \chi^{(\lambda)}_{(l)} s^{(l)} \]
and this again proves the equality of the two inner products. In other words, the Frobenius’ formula provides the unitary transformation connecting the boson and fermion bases in each energy $E = |\{\lambda\}| = n$ subspace. The transformation is unitary because of the orthogonality of the characters of the permutation group $S_n$.

**Miwa variables and $\tau$-functions**

We expressed the $h_n$ in terms of the $s_n$ by using the formula

$$\exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} s_n(z) t^n\right\} = \sum_{n=0}^{\infty} h_n(z) t^n.$$  

The works of Sato, Miwa and Jimbo rescale the $s_n$ by setting

$$x_n = \frac{1}{n} s_n.$$  

These $x_n$ are usually called *Miwa variables*. We will soon see that they are to be identified with the multi-time variables $x_n$.

The formula for the $h_n$ now becomes

$$\exp\left\{\sum_{n=1}^{\infty} x_n t^n\right\} = \sum_{n=0}^{\infty} h_n t^n.$$  

By differentiating each side of the definition we find that

$$\frac{\partial}{\partial x_m} h_n(x) = h_{n-m}(x).$$  

For example

$$h_{-n}(x) = 0,$$

$$h_0(x) = 1,$$

$$h_1(x) = x_1,$$

$$h_2(x) = x_2 + \frac{1}{2} x_1^2,$$

$$h_3(x) = x_3 + x_1 x_2 + \frac{1}{6} x_1^3.$$  

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We can combine these with the Jacobi-Trudi formulæ for the Schur functions

\[ \Psi_1 = h_1, \]
\[ \Psi_2 = h_2, \]
\[ \Psi_3 = h_1^2 - h_2, \]
\[ \Psi_4 = h_1h_2 - h_3, \]

\[ \Psi_6 = h_1h_2 - h_3, \]

to get expressions for the Schur functions in terms of the \( x_n \). There is therefore a one-to-one correspondence between the charge neutral excited states and functions of the \( x_n \). These functions are called \( \tau \)-functions. Let

\[ |\text{gnd}\rangle = |\ldots, 0, 0, 0 : 1, 1, 1, \ldots \rangle \]

be the state with all the negative energy states occupied. The state immediately to the right of the “:” is \( z^{N-1} \), and the empty state to the left is \( z^N \). We will renormalize their momenta to \( n = -1 \) and \( n = 0 \) respectively. Then

\[
\begin{align*}
|\ldots, 0, 0, 1 : 0, 1, 1, \ldots \rangle & \leftrightarrow \tau_{(1)}(x) = x_1, \\
|\ldots, 0, 1, 0 : 0, 1, 1, \ldots \rangle & \leftrightarrow \tau_{(2)}(x) = x_2 + \frac{1}{2}x_1^2, \\
|\ldots, 0, 0, 1 : 1, 0, 1, \ldots \rangle & \leftrightarrow \tau_{(12)}(x) = -x_2 + \frac{1}{2}x_1^2, \\
|\ldots, 1, 0, 0 : 0, 1, 1, \ldots \rangle & \leftrightarrow \tau_{(3)}(x) = x_3 + x_1x_2 - 2 + \frac{1}{6}x_1^3, \\
|\ldots, 0, 1, 0 : 1, 0, 1, \ldots \rangle & \leftrightarrow \tau_{(21)}(x) = -x_3 + \frac{1}{2}x_1^3,
\end{align*}
\]

where

\[
\begin{align*}
|\ldots, 0, 0, 1 : 0, 1, 1, \ldots \rangle &= (\hat{\psi}^+_0 \hat{\psi}^{-}_1)|\text{gnd}\rangle, \\
|\ldots, 0, 1, 0 : 0, 1, 1, \ldots \rangle &= (\hat{\psi}^+_1 \hat{\psi}^{-}_1)|\text{gnd}\rangle, \\
|\ldots, 0, 0, 1 : 1, 0, 1, \ldots \rangle &= (\hat{\psi}^+_0 \hat{\psi}^{-}_1)(\hat{\psi}^+_1 \hat{\psi}^{-}_2)|\text{gnd}\rangle, \\
|\ldots, 1, 0, 0 : 0, 1, 1, \ldots \rangle &= (\hat{\psi}^+_2 \hat{\psi}^{-}_1)|\text{gnd}\rangle, \\
|\ldots, 0, 1, 0 : 1, 0, 1, \ldots \rangle &= (\hat{\psi}^+_1 \hat{\psi}^{-}_1)(\hat{\psi}^+_2 \hat{\psi}^{-}_2)|\text{gnd}\rangle.
\end{align*}
\]

Here the \( \hat{\psi}_i, \hat{\psi}^+_i \) are the operators that create and annihilate a fermion in the single-particle state with wavefunction \( z^{N+i} \), they obey

\[
\{\hat{\psi}_i, \hat{\psi}^+_j\} = \delta_{ij}, \quad \{\hat{\psi}^+_i, \hat{\psi}^+_j\} = \{\hat{\psi}_i, \hat{\psi}_j\} = 0.
\]
The ground state obeys
\[ \hat{\psi}_n |\text{gnd}\rangle = 0, \quad n \geq 0, \]
and
\[ \hat{\psi}_n^\dagger |\text{gnd}\rangle = 0, \quad n < 0. \]
We can also set
\[ |\text{gnd}\rangle = \hat{\psi}_1^\dagger \hat{\psi}_2^\dagger \cdots \hat{\psi}_N^\dagger |0\rangle \]
where \( |0\rangle \) is the vacuum, or no-particle, state.

Now we introduce the Fermi fields
\[ \hat{\Psi}(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}_n z^n, \quad \hat{\Psi}(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}_n^\dagger z^{-n}. \]
that act at \( z = e^{i\theta} \) to respectively annihilate or create a fermion there. In terms of them the many-body wavefunction corresponding to the state \( |\psi\rangle \) is
\[ \psi(z_1, \ldots, z_N) = \langle 0 | \hat{\Psi}(z_N) \cdots \hat{\Psi}(z_1) |\psi\rangle \times \prod_{i=1}^{N} z_i^N. \]
The last factor accounts for the shift in the origin of the momentum quantum number. In particular the \( N \)-particle ground-state wavefunction is given by
\[ \Psi_0(z_1, \ldots, z_N) = \langle 0 | \hat{\Psi}(z_N) \cdots \hat{\Psi}(z_1) \hat{\psi}_1^\dagger \hat{\psi}_2^\dagger \cdots \hat{\psi}_N^\dagger |0\rangle \times \prod_{i=1}^{N} z_i^N. \]

There is another way of relating a state \( |\chi\rangle \) to a \( \tau \)-function. This the way in which the \( \tau \)-functions appear in the works of Sato and his school. Their approach essentially uses the very useful formula in the form
\[ \exp \left\{ \sum_{n=1}^{\infty} x_n s_n(z) \right\} = \sum_{\{\lambda\}} \Phi_{\{\lambda\}}(x_n) \Phi_{\{\lambda\}}(z_i), \]
where the first Schur function on the right has been expressed in terms of Miwa variables \( x_n \). These variables now have no associated underlying \( z_i \)'s, but are mere parameters. Indeed, the fermion co-ordinates \( z_i \) never appear in the Sato-school papers.
We now define the Slater-determinant wavefunction
\[
\Psi_{(x_1,x_2,\ldots)}(z) \overset{\text{def}}{=} \exp \left\{ \sum_{n=1}^{\infty} x_n s_n(z) \right\} \Psi_0(z) = \det \exp \left\{ \sum_n x_n z_n s_n \right\} z_s^{N-t} \]
and identify it with the bosonic coherent state
\[
| x_1, x_2, x_3, \ldots \rangle = \exp \left\{ \sum_{n=1}^{\infty} x_n \hat{a}_n^\dagger \right\} | \text{gnd} \rangle.
\]
Then we can write
\[
\tau(x) = \langle \chi | x_1, x_2, \ldots \rangle = \langle \chi | \exp \left\{ \sum_{n=1}^{\infty} x_n \hat{a}_n^\dagger \right\} | \text{gnd} \rangle.
\]
We end up with the same one-to-one mapping between fermion states and polynomials in \( x_n \) as before, although now the fermion state involves matrix elements of the dual-space vector \( \langle \chi \rangle \), and so is antilinear. We will write this formula as
\[
\tau(\chi) = \langle \chi | \exp \{ H(x) \} | \text{gnd} \rangle
\]
where
\[
H(x) = \sum_{n=1}^{\infty} x_n \hat{a}_n^\dagger.
\]

Now we consider the effect of the field operators on \( \langle \chi \rangle \). The operator \( \hat{\Psi}(z) \) acts to its left to create a particle at \( z \), and \( \hat{\Psi}^\dagger(z) \) to destroy one. Let \(| + 1 \rangle = \psi_0^\dagger | \text{gnd} \rangle \) and \(| - 1 \rangle = \psi_1^\dagger | \text{gnd} \rangle \), then we can represent these left actions by vertex operators. I claim that, for \( | \chi \rangle \) an \( N \)-particle state, we have
\[
\langle \chi | \hat{\Psi}(z) e^{H(x)} | + 1 \rangle = \exp \left\{ \sum_{n=1}^{\infty} x_n z^n \right\} \exp \left\{ -\sum_{n=1}^{\infty} z^{-n} \frac{1}{n} \frac{\partial}{\partial x_n} \right\} \tau(\chi)(x_1, x_2, \ldots),
\]
and
\[
\langle \chi | \hat{\Psi}^\dagger(z) e^{H(x)} | - 1 \rangle = z \exp \left\{ -\sum_{n=1}^{\infty} x_n z^n \right\} \exp \left\{ \sum_{n=1}^{\infty} z^{-n} \frac{1}{n} \frac{\partial}{\partial x_n} \right\} \tau(\chi)(x_1, x_2, \ldots).
\]
Granted these formula, we can immediately see where the bilinear relation comes from. We take a tensor product $H^* \otimes H^*$ of two copies of the dual of the many-fermion Hilbert space, and consider the action on it of the operator

$$
\hat{Q} = \sum_n \hat{\psi}_n \otimes \hat{\psi}_n^\dagger = \frac{1}{2\pi} \int_0^{2\pi} \hat{\Psi}(z) \otimes \hat{\Psi}^\dagger(z) \, d\theta = \frac{1}{2\pi i} \oint \hat{\Psi}(z) \otimes \hat{\Psi}^\dagger(z) \frac{dz}{z}.
$$

Applied to $\langle \text{gnd} | \otimes | \text{gnd} \rangle$ we have

$$
(\langle \text{gnd} | \otimes | \text{gnd} \rangle) \hat{Q} = 0
$$

because one of the factors in each term in the sum over $n$ acts as an annihilation operator. Now $\hat{Q}$ is unchanged if we make a unitary transformation

$$
\hat{\psi}_n \rightarrow \hat{\psi}_m U_{mn}, \quad \hat{\psi}_n^\dagger \rightarrow U_{mn}^* \hat{\psi}_m^\dagger,
$$

and so $\hat{Q}$ will kill any state of the form $\langle \chi | \otimes | \chi \rangle$ where $| \chi \rangle$ can be represented by a single Slater determinant. Thus if we define

$$
w(x, z) = \frac{\langle \chi | \hat{\Psi}(z)e^{H(x)}| + 1 \rangle}{\langle \chi | e^{H(x)}| \text{gnd} \rangle} = \tau \left( x_1 - \frac{1}{z}, x_2 - \frac{1}{2z^2}, x_3 - \frac{1}{3z^3}, \ldots \right) \exp\{\xi(x, z)\} / \tau(x_1, x_2, x_3, \ldots)
$$

and

$$
w^*(x, z) = \frac{1}{z} \frac{\langle \chi | \hat{\Psi}^\dagger(z)e^{H(x)}| - 1 \rangle}{\langle \chi | e^{H(x)}| \text{gnd} \rangle} = \tau \left( x_1 + \frac{1}{z}, x_2 + \frac{1}{2z^2}, x_3 + \frac{1}{3z^3}, \ldots \right) \exp\{-\xi(x, z)\} / \tau(x_1, x_2, x_3, \ldots)
$$

we have

$$
\frac{1}{2\pi i} \oint w(x, z)w^*(x, z) \, dz = \text{Res}_z \{w(x, z)w^*(x, z)\} = 0
$$

for the $\tau$-function of any single-Slater state $\langle \chi |$.

**Proof of Vertex-Operator Identities**
Start with the \((N + 1)\)-particle ground state wavefunction, and let \(\hat{\Psi}(z)\) act to annihilate one of the particles. The resulting \(N\)-particle wavefunction is
\[
\langle 0 | \hat{\Psi}(z_N) \cdots \hat{\Psi}(z_1) \hat{\psi}_0^\dagger \cdots \hat{\psi}_{-N}^\dagger | 0 \rangle \times \prod_{i=1}^{N} z_i^N
\]
\[
= \frac{1}{z^N} \prod_{i=1}^{N} (z - z_i) \prod_{i<j} (z_i - z_j)
\]
\[
= \prod_{i=1}^{N} \left(1 - \frac{z_i}{z}\right) \Psi_0(z_1, \ldots, z_N)
\]
\[
= \exp \left\{ -\sum_{n=1}^{\infty} \frac{s_n(z_i)}{nz^n} \right\} \Psi_0(z_1, \ldots, z_N).
\]
Thus
\[
\langle 0 | \hat{\Psi}(z_N) \cdots \hat{\Psi}(z_1) \left( \hat{\psi}(z) | + 1 \right) \times \prod_{i=1}^{N} z_i^N = \exp \left\{ -\sum_{n=1}^{\infty} \frac{s_n(z_i)}{nz^n} \right\} \Psi_0(z_1, \ldots, z_N).
\]
Similarly we start with the \((N - 1)\)-particle wavefunction and let \(\hat{\psi}_1^\dagger(z)\) create another particle. The resulting \(N\)-particle wavefunction is
\[
\langle 0 | \hat{\Psi}(z_N) \cdots \hat{\Psi}(z_1) \hat{\psi}_1^\dagger(z) \hat{\psi}_{-2}^\dagger \cdots \hat{\psi}_{-N}^\dagger | 0 \rangle \times \prod_{i=1}^{N} z_i^N
\]
\[
= \langle 0 | \hat{\Psi}(z_N) \cdots \hat{\Psi}(z_1) \sum_{n=-1}^{\infty} \hat{\psi}_n^\dagger z^{-n} \hat{\psi}_{-2}^\dagger \hat{\psi}_{-3}^\dagger \cdots \hat{\psi}_{-N}^\dagger | 0 \rangle \times \prod_{i=1}^{N} z_i^N
\]
\[
= z \sum_{m=0}^{\infty} \left| \begin{array}{cccc}
\sum_{n=-1}^{\infty} z_i^{N-1+2m} z^{-1} z^{N-2} z^{-m} & \cdots & 1 \\
\vdots & \ddots & \vdots & \vdots \\
\sum_{n=-1}^{\infty} z_i^{N-1+2m} z^{N-2} z^{-m} & \cdots & 1 \\
\sum_{n=-1}^{\infty} z_i^{N-1+2m} z^{N-2} z^{-m} & \cdots & 1 \\
\end{array} \right|
\]
\[
= z \left( \sum_{m=0}^{\infty} h_m(z_i) z^{-m} \right) \Psi_0(z_1, \ldots, z_N)
\]
\[
= z \exp \left\{ \sum_{n=1}^{\infty} \frac{s_n(z_i)}{nz^n} \right\} \Psi_0(z_1, \ldots, z_N).
\]
\[ \langle 0 | \hat{\Psi}(z_N) \cdots \hat{\Psi}(z_1) \left( \hat{\Psi}^\dagger(z) - 1 \right) \times \prod_{i=1}^{N} z_i^N = \exp \left\{ \sum_{n=1}^{\infty} \frac{s_n(z_i)}{n z^n} \right\} \Psi_0(z_1, \ldots, z_N). \]

Now we also have that
\[
\hat{\Psi}(z) \exp \{ H(x) \} = \exp \{ H(x) \} \hat{\Psi}(z) e^{\xi(x,z)},
\]
\[
\hat{\Psi}^\dagger(z) \exp \{ H(x) \} = \exp \{ H(x) \} \hat{\Psi}^\dagger(z) e^{-\xi(x,z)}.
\]

These formulæ hold because, depending on whether we we kill or create the fermion at \( z = e^{i\theta} \) before or after the multiplication by the \( s_n(z_i) \), there is, or is not, a term involving \( \sum x z^n \) included in the \( \sum x_n s_n \).

Consequently
\[
\langle \chi | \hat{\Psi}(z) \exp \{ H(x) \} | + 1 \rangle = \langle \chi | \exp \{ H(x) \} \hat{\Psi}(z) | + 1 \rangle e^{\xi(x,z)}
\]
\[
= \langle \chi | \exp \left\{ H \left( x_n - \frac{1}{n z^n} \right) \right\} | \text{gnd} \rangle e^{\xi(x,z)}
\]
\[
= e^{\xi(x,z)} \tau_{\langle \chi |} \left( x_1 - \frac{1}{z}, x_2 - \frac{1}{2 z^2}, \ldots \right)
\]
\[
= e^{\xi(x,z)} \exp \left\{ - \sum_{n=1}^{\infty} \frac{1}{n z^n} \frac{\partial}{\partial x_n} \right\} \tau_{\langle \chi |} (x_1, x_2, \ldots).
\]

and
\[
\langle \chi | \hat{\Psi}^\dagger(z) \exp H(x) | - 1 \rangle = \langle \chi | \exp H(x) \hat{\Psi}^\dagger(z) | - 1 \rangle e^{\xi(x,z)}
\]
\[
= z \langle \chi | \exp H \left( x_n + \frac{1}{n z^n} \right) | \text{gnd} \rangle e^{-\xi(x,z)}
\]
\[
= ze^{-\xi(x,z)} \tau_{\langle \chi |} \left( x_1 + \frac{1}{z}, x_2 + \frac{1}{2 z^2}, \ldots \right)
\]
\[
= ze^{-\xi(x,z)} \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{n z^n} \frac{\partial}{\partial x_n} \right\} \tau_{\langle \chi |} (x_1, x_2, \ldots).
\]