Kertész Line in the Three-Dimensional Compact U(1) Lattice Higgs Model

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(Received 24 March 2005; published 26 July 2005)

The three-dimensional lattice Higgs model with compact U(1) gauge symmetry and unit charge is investigated by means of Monte Carlo simulations. The full model with fluctuating Higgs amplitude is simulated, and both energy as well as topological observables are measured. The data show a Higgs and a confined phase separated by a well-defined phase boundary, which is argued to be caused by proliferating vortices. For fixed gauge coupling, the phase boundary consists of a line of first-order phase transitions at small Higgs self-coupling, ending at a critical point. The phase boundary then continues as a Kertész line across which thermodynamic quantities are nonsingular. Symmetry arguments are given to support these findings.

DOI: 10.1103/PhysRevLett.95.051601 PACS numbers: 11.15.Ha, 12.38.Aw

Being one of the few well-understood theories exhibiting charge confinement, the three-dimensional (3D) pure compact U(1) gauge theory plays a central role in the study of deconfinement transitions [1]. An intriguing extension is obtained by coupling a scalar matter field to this confining gauge theory [2]. The resulting Higgs model and its extensions have recently attracted considerable attention also in the condensed matter community as effective descriptions of quantum critical phenomena [3]. In a seminal paper, Fradkin and Shenker [2] studied the phase diagram of the model in the London limit, where the Higgs field has a fixed amplitude. They concluded that in the case of a Higgs field carrying one unit charge $q = 1$, it is always possible to move from the Higgs region into the confined region without encountering singularities in local gauge-invariant observables. As for the liquid-vapor transition, this is commonly interpreted as implying that the two ground states do not constitute distinct phases. This is supported further by symmetry considerations [4]. In the absence of matter fields, the 3D pure noncompact U(1) gauge theory is characterized by a *global* magnetic U(1) symmetry. When magnetic monopoles are introduced, which are pointlike instanton solutions of the *compact* U(1) gauge theory, this global symmetry becomes anomalous and only the discrete subgroup Z of integer numbers survives. When the compact gauge theory is subsequently coupled to a Higgs field carrying charge q, the magnetic symmetry is further reduced to the finite cyclic subgroup Z_q of q elements. For $q = 2$, this recovers the known result that the model undergoes a continuous phase transition belonging to the 3D Ising universality class [2,5]. For $q = 1$, this argument excludes a continuous phase transition because the group Z_1, consisting of only the unit element, cannot be spontaneously broken.

In this Letter, we argue that Monte Carlo data show a more refined picture with two distinct phases separated by a well-defined phase boundary. We consider the $q = 1$ model with fluctuating Higgs amplitude, as was done first in Refs. [6,7] on smaller lattices, and more recently in Refs. [8,9] on larger ones. The ensuing picture, which turns out to be closely related to the dual superconductor mechanism of confinement [10], essentially vindicates the scenario put forward by Einhorn and Savit [11], in which the transition from the Higgs to the confined phase is triggered by proliferating vortices. As discussed below, the nature of this mechanism is consistent with the Fradkin-Shenker result [2] and the symmetry argument [4]. Our results are at odds with the vortex string-breaking scenario put forward by Nagaosa and Lee [12], who recently argued that only the confined phase is present in the compact theory. Furthermore, we found no support for a deconfinement phase transition of the Berezinsky-Kosterlitz-Thouless (BKT) type recently proposed in Ref. [13], as for sufficiently large lattices we observe no scaling behavior at all across the phase boundary where the BKT transition is supposed to arise.

The compact U(1) Higgs model is specified by the Euclidean lattice action $S = S_g + S_\phi$, with the gauge part

$$S_g = \beta \sum_{x,\mu<\nu} [1 - \cos\theta_{\mu\nu}(x)]. \tag{1}$$

Here, β is the inverse gauge coupling, the sum extends over all lattice sites x and lattice directions μ, and $\theta_{\mu\nu}(x)$ denotes the plaquette variable $\theta_{\mu\nu}(x) = \Delta_\mu\theta_\nu(x) - \Delta_\nu\theta_\mu(x)$, with the lattice derivative $\Delta_\nu\theta_\mu(x) \equiv \theta_\mu(x+\nu) - \theta_\mu(x)$ and the compact link variable $\theta_\mu(x) \in [-\pi, \pi)$. The matter part of the action S is given by

$$S_\phi = -\kappa \sum_{x,\mu} \rho(x)\rho(x+\mu)\cos[\Delta_\mu\varphi(x) - q\theta_\mu(x)] + \sum_x \{\rho^2(x) + \lambda[\rho^2(x) - 1]^2\}, \tag{2}$$

where polar coordinates are chosen to represent the complex Higgs field $\phi(x) = \rho(x)e^{i\varphi(x)}$, with $\varphi(x) \in [-\pi, \pi)$, κ is the hopping parameter, and λ the Higgs self-coupling. The pure $|\phi|^4$ theory with fluctuating amplitude, obtained by taking the limit $\beta \to \infty$, was recently investigated by means of Monte Carlo simulations in Ref. [14]. We consider the system in three spacetime dimensions, taking one of the dimensions to represent (Euclidean) time.

The precise nature of the phase diagram is investigated numerically by studying several (gauge-invariant) observables chosen such that the gauge and matter parts are probed separately. Following Ref. [15], where the London limit of the model was considered, the gauge part is studied by measuring the monopole density M, as defined in Ref. [16], and the Polyakov loop. Both observables distinguish a confined from a deconfined phase. The matter part of the model is studied by measuring the Higgs amplitude squared $\rho^2 \equiv (1/L^3)\Sigma_x \rho^2(x)$, where L is the linear size of the cubic lattice. This bulk operator distinguishes the Higgs phase from a disordered one. In addition, the plaquette action (1) (divided by $3L^3$) and the link observable

$$C = -\frac{1}{3L^3}\sum_{x,\mu} \cos[\Delta_\mu \varphi(x) - q\theta_\mu(x)] \qquad (3)$$

are monitored. Both Metropolis and heat-bath methods were used to generate Monte Carlo updates. Since these updates become inefficient in regions of first-order phase transitions, the multicanonical method [17] and reweighting techniques [18] were implemented to access these regions of phase space. The simulations were carried out at fixed β on cubic lattices varying in size from 6^3 to 32^3, in extreme cases to 42^3. Thermalization of the production runs typically took 4×10^4 sweeps of the lattice, while about 10^6 sweeps were used to collect data, with measurements taken after each sweep of the lattice. The maxima of the link susceptibility $\chi_C = L^3(\langle C^2\rangle - \langle C\rangle^2)$ and histograms rescaled to equal height have been used to determine the location of the phase boundary. We have chosen the link susceptibility to trace out the phase diagram because its peaks are more pronounced than for the other observables. We have checked that within the achieved accuracy, the monopole susceptibility peaks at the same location as χ_C does. Statistical errors were estimated by means of jackknife binning. For a detailed description of the algorithms and their implementation, see Ref. [19].

The phase diagram [6,7,9], summarized in Fig. 1 for fixed gauge coupling $\beta = 1.1$, consists of two separate phases: a confined and a Higgs phase. In the lower right part of the phase diagram, the average Higgs amplitude squared takes on a minimum value (see Fig. 2). The monopole density is finite here and practically independent of κ and λ. Snapshots of monopole configurations (see bottom inset in Fig. 1) show that the monopoles are in the plasma phase. As β increases, the monopole density decreases. The monopoles become completely suppressed in the weak gauge coupling limit $\beta \to \infty$, where the model reduces to the pure $|\phi|^4$ theory. The average plaquette action takes on a value also practically independent of κ and λ. These observables signal that electric charges are confined. This *confined phase* persists in the limit $\kappa \to 0$, where the model reduces to the pure compact U(1) gauge theory first studied by Polyakov [1].

In the upper left part of the phase diagram, the average $\langle\rho^2\rangle$ increases more or less linearly with increasing κ (note the logarithmic scale used for ρ^2 in Fig. 2). The monopole density is vanishing small here and the few monopoles still present are tightly bound in monopole-antimonopole pairs [15] (see top inset in Fig. 1). Being rendered ineffective, the monopoles can no longer confine electric charges. Taken together, these observables identify this phase as

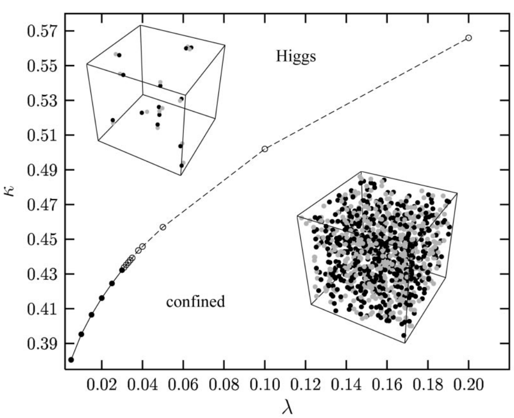

FIG. 1. Phase diagram of the model at $\beta = 1.1$ in the infinite-volume limit. Solid dots mark the location of a line of first-order phase transitions. This line ends at a critical point around $0.031 < \lambda_c < 0.032$. Open dots for $\lambda > \lambda_c$ mark the location of the Kertész line (see text), approaching $\kappa = 0.717(2)$ in the London limit $\lambda \to \infty$. Statistical error bars are smaller than the symbol size in the figure. The insets show snapshots of typical monopole configurations in both phases, with black dots denoting monopoles and gray dots denoting antimonopoles.

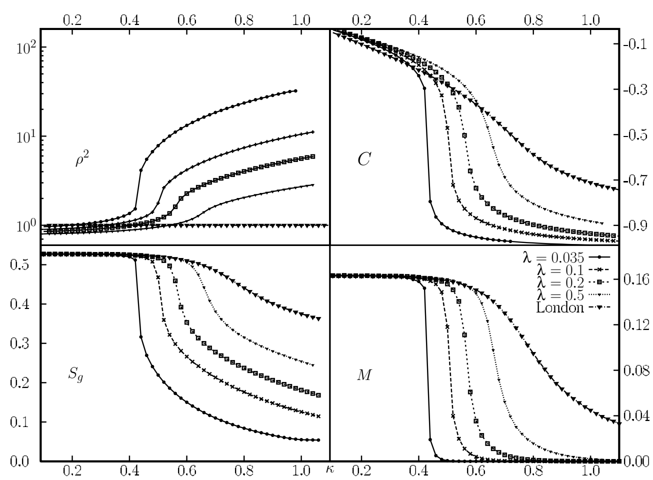

FIG. 2. Averages of the Higgs amplitude squared ρ^2, link observable C, plaquette action S_g, and monopole density M as a function of κ for various values of the self-coupling λ, including the London limit $\lambda \to \infty$, at $\beta = 1.1$ on a relatively small lattice ($L = 12$).

Higgs phase. The identification of the two phases agrees with the behavior of the Polyakov loop we observed.

We next examine how for given inverse gauge coupling β the confined phase goes over into the Higgs phase. Below a critical point $\lambda_c(\beta)$, we observe metastable behavior typical for first-order phase transitions, in accord with earlier Monte Carlo results obtained on smaller lattices [6,7]. Simulations for different values of $\beta \in [1.1, 2.0]$ show that the first-order phase transition becomes more pronounced for strong gauge coupling, i.e., small β. At fixed β, the transition becomes stronger with decreasing λ, where fluctuations in the Higgs amplitude become more volatile [6]. For each value of λ considered, the model was simulated on lattices of different sizes to study finite-size effects and to obtain precise estimates of the location of the first-order phase transitions, using the multicanonical approach and reweighting techniques. The first-order line ends at a critical point, which for $\beta = 1.1$ and infinite volume we estimated to be located in the interval $0.030 < \lambda_c < 0.032$. We have not attempted to establish the nature of the critical point, as critical slowing down requires very long runs of our code based on locale updates.

Above λ_c, we observe the same remarkable behavior as previously found in the London limit $\lambda \rightarrow \infty$, where fluctuations in the Higgs amplitude are completely frozen [5,15]. Namely, for sufficiently large lattices, the maxima of the susceptibilities do not show any finite-size scaling (see Fig. 3) and the susceptibility data for the observables in Fig. 2 obtained on different lattice sizes collapse onto single curves without rescaling, indicating that the infinite-volume limit is reached. Since a first-order phase transition can be excluded in this region, the absence of finite-size scaling suggests the absence of thermodynamic singularities in the infinite-volume limit. To also exclude a crossover in the usual sense, we have checked that the maxima of χ_C do not depend on the direction in which the phase boundary is crossed, either by varying λ, or κ [20]. This analysis is performed using multihistogram reweighting techniques to achieve a high accuracy in determining the peak locations. To sum up this part, despite the presence of a well-defined and precisely located phase boundary, no ordinary phase transition or crossover in the usual sense seem to come into question above λ_c.

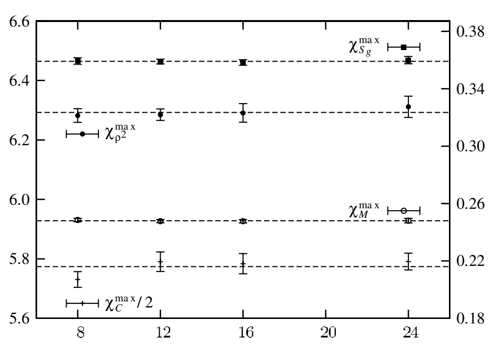

FIG. 3. Maxima in the susceptibilities of the plaquette action S_g, Higgs amplitude squared ρ^2, monopole density M, and link variable C as a function of the lattice size L at $\beta = 1.1$ and $\lambda = 0.2$. To fit all the data in one figure, $\frac{1}{2}\chi_C^{\max}$ rather than $\chi_C^{\max}$ is plotted.

In the dual superconductor scenario of confinement [10], monopoles are pictured in the Higgs phase as being tightly bound together in monopole-antimonopole pairs, just as we observed (see top inset in Fig. 1). The magnetic flux emanating from a monopole is squeezed into a short flux tube, or vortex, carrying one unit $2\pi/q$ ($q = 1$) of magnetic flux, which terminates at an antimonopole. The vortices, which in this phase can also exist as small fluctuating loops, have a finite line tension. Upon approaching the phase boundary, the vortex line tension vanishes. At this point, the vortices proliferate, gaining configurational entropy without energy cost, and an infinite vortex network appears which disorders the Higgs ground state. At the same time, the monopoles are no longer bound in tight monopole-antimonopole pairs but form, as seen in the bottom inset in Fig. 1, a plasma which exhibits charge confinement. This scenario explains the existence of a well-defined phase boundary separating the Higgs and confined phases [11]. The proliferation of vortices persists in the weak gauge coupling limit $\beta \rightarrow \infty$, corresponding to the pure $|\phi|^4$ theory with global U(1) symmetry. Only in this limit, the proliferating vortices cause a continuous phase transition, belonging to the *XY* universality class in which the spontaneously broken global U(1) symmetry is restored. Outside this limit, in the absence of a relevant global symmetry, the proliferating vortices do not lead to singularities in thermodynamic quantities and are not connected to a symmetry breaking transition.

The situation is reminiscent of the Ising model in an external magnetic field. It was shown by Fortuin and Kasteleyn (FK) [21] that for zero field this spin model and its thermal critical behavior can be equivalently formulated as a correlated percolation problem by putting bonds between nearest neighbor spins in the same spin state. The bonds are set with a temperature-dependent probability $p_{\rm FK}(T)$. The clusters thus constructed percolate precisely at the Curie point and have the Ising critical exponents encoded in their fractal structure. By applying an external field, one explicitly breaks the global Z_2 symmetry of the Ising model, and the partition function becomes analytic in temperature, excluding a thermal phase transition. Yet, for a given applied field H, the FK clusters still percolate at a precisely defined temperature $T_{\rm p}(H)$. The resulting percolation line in the phase diagram is known as the *Kertész line* [22]. Although percolation observables remain singular along the line, no thermodynamic singularities are encountered when crossing it [22,23]. In the limit $H \rightarrow \infty$, all the spins are aligned along the field, so that the FK construction reduces to random bond percolation. The Kertész line therefore ends at the temperature determined by $p_{\mathrm{FK}}(T_{\mathrm{p}}) = p_{\mathrm{c}}$, with p_{c} denoting the random bond percolation threshold. Along the entire Kertész line, the percolation observables have the usual percolation exponents.

The vortex proliferation line in the 3D compact U(1) Higgs model is the analog of the Kertész line in the Ising model in an external field (for a related discussion of the 4D model, see Ref. [24]). Such an interpretation of a deconfinement transition as a Kertész line was first proposed in the context of the SU(2) Higgs model [25–29]. The similarity with the Ising model can be made more precise by considering the London limit $\lambda \to \infty$ of the compact U(1) Higgs model. In this limit, the vortex proliferation line starts at the XY critical point $\kappa = \kappa_{XY}$, $\beta = \infty$. To identify the end point, we note that for $\kappa \to \infty$, the model reduces to a Z_q gauge theory. In 3D, such a discrete gauge theory is dual to the q-state Potts model with *global* Z_q symmetry, which undergoes a phase transition at some critical value β_q of the inverse gauge coupling [30]. Since the limit $q \to 1$ of the Potts model describes random bond percolation, we conclude that the vortex proliferation line ends in a random bond percolation critical point at $\beta = \beta_1$, $\kappa = \infty$. The vortex network present in the vicinity of this critical point is expected to be similar to the one studied in the context of the Kibble mechanism for cosmic string formation [31], which was shown to belong to the random percolation universality class [32]. By continuity, we expect that, although not connected to thermodynamic singularities, the percolation observables have random percolation exponents along the entire vortex proliferation line. In the limit $\beta \to \infty$, these exponents are expected to cross over to the ones appropriate for the XY universality class. In a future study, we plan to investigate the vortex network using percolation observables to numerically verify these conjectures directly. Such a study is more complicated than the indirect study of the vortex network presented here, via monopoles, which involves only the gauge sector of the theory and has the advantage that the relevant observables (monopole density and corresponding susceptibility) can, in contrast to vortex percolation observables [33,34], be defined unambiguously.

In conclusion, our Monte Carlo data on the 3D compact U(1) lattice Higgs model show that the Higgs and confined phases are separated by a well-defined phase boundary due to proliferating vortices. For fixed gauge coupling, the phase boundary is a line of first-order phase transitions at small Higgs self-coupling, which ends at a critical point. The phase boundary then continues as a Kertész line across which thermodynamic quantities and other local gauge-invariant observables are nonsingular. In the London limit, the Kertész line defined by the proliferating vortices connects the XY and random bond percolation critical points, which both form limiting cases of the compact U(1) Higgs model.

This work is partially supported by the DFG Grant No. JA 483/17-3. S.W. acknowledges financial support from the Studienstiftung des deutschen Volkes.

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