Large-order perturbation theory of the Zeeman effect in hydrogen from a four-dimensional anisotropic anharmonic oscillator

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The Zeeman Hamiltonian for (spinless) hydrogen in a constant magnetic field is shown to be equivalent to a four-dimensional anisotropic anharmonic oscillator. Using this relation, Rayleigh-Schrödinger perturbation series expansions of both systems can be related to each other and analyzed in a unified way. Special emphasis is laid upon analytical estimates of their behavior in large orders of perturbation theory. Employing the path-integral approach, a new large-order formula is derived for the expansion of the ground-state energy of the oscillator system. With use of known Bender-Wu formulas for isotropic anharmonic oscillators, the major part of this calculation becomes straightforward. Combined with the above equivalence, this calculation represents the simplest path-integral derivation of large-order formulas for the Zeeman system.

I. INTRODUCTION

The Zeeman effect, describing atoms in constant magnetic field, was one of the earliest problems studied[1] in quantum mechanics. For spinless hydrogen (with infinitely heavy nucleus), the Hamiltonian is given by

$$H = \tfrac{1}{2}(\mathbf{p}-\mathbf{A})^2 - 1/r = \tfrac{1}{2}\mathbf{p}^2 - 1/r + \tfrac{1}{8}B^2(x^2+y^2) + \tfrac{1}{2}BL_z\,, \qquad (1)$$

where $\mathbf{p} = -i\nabla$, $\mathbf{A} = \frac{1}{2}(\mathbf{B}\times\mathbf{r})$ is the vector potential, and L_z is the angular momentum in the z direction, which is taken along the magnetic field direction, $\mathbf{B} = B\mathbf{e}_z$. In the natural units used in (1), $B = 1$ corresponds to $\frac{1}{2}mc^2\alpha^2/\mu_B = 2.35\times10^9$ G, which is extremely large compared with typical laboratory fields around 10^4 G.

Although many numerical and analytical investigations of this Hamiltonian have been reported in the literature,[2] there is still today considerable interest in its detailed properties, especially in strong magnetic fields.[3] The main motivation comes from astrophysics[4,5] where very strong magnetic fields are needed in the description of neutron star surfaces (10^{10}–10^{13} G) and white dwarf stars (10^7–10^8 G). Also in some solid-state systems *effective* magnetic fields may reach up to 10^{10} G.[5]

About a decade ago, much effort went into deriving formulas for the large-order coefficients in the Rayleigh-Schrödinger perturbation series expansions for the Zeeman energies.[6–14] All these expansions turn out to be asymptotic series. For the ground-state energy of (1), $E^{(0)} = -\frac{1}{2} + \sum_{k=1} E_k (B^2/8)^k$, the result is[9]

$$E_k \underset{k\to\infty}{\sim} -\left(\frac{4}{\pi}\right)^{5/2}\left(-\frac{8}{\pi^2}\right)^k \Gamma\left(2k+\frac{3}{2}\right)\,. \qquad (2)$$

Subsequently, this information was used to apply efficient resummation algorithms to these asymptotic series, allowing accurate computations of the energies, even for strong fields.[15]

Based on Schrödinger's formulation of quantum mechanics, the large-order formulas have been derived by means of quite involved multidimensional WKB techniques applied to the differential operator (1). These calculations can be considered as a nontrivial generalization of the original derivations[16–18] of the corresponding Bender-Wu formulas for anharmonic oscillators. While the latter have been rederived later from path-integral approaches,[19,20] which are conceptually more transparent and technically simpler, such approaches were never taken for the Zeeman system.

The purpose of this paper is to present such a path-integral derivation, albeit in a somewhat indirect fashion. The motivation to consider this problem once more comes from a property of the Zeeman system which has apparently been overlooked so far, namely that the Zeeman Hamiltonian (1) in *three* dimensions is equivalent to an anisotropic anharmonic oscillator in *four* dimensions. By applying path-integral methods to the latter, this allows a very transparent and simple derivation of large-order formulas for the Zeeman system. A brief account of this approach has been given elsewhere.[21]

That it is in principle possible to relate three-dimensional Coulomb systems to four-dimensional oscillators has been known ever since the early work of Schrödinger[22] on his wave equation in 1941. Later such an equivalence was rediscovered by Kustaanheimo and Stiefel[23] in the context of the classical Kepler problem in celestial mechanics. More recently, the latter formulation was used by Duru and Kleinert[24] to obtain the path-integral quantization of the pure Coulomb

potential in terms of a harmonic oscillator. Another important example is the Stark Hamiltonian for a hydrogen atom in constant electric field, which can be shown[14,17(b),25] to be equivalent to two decoupled two-dimensional isotropic oscillators with quartic anharmonicity. Making use of known results for anharmonic oscillators, this allowed a very efficient calculation of perturbation expansions for Stark resonances and their large-order behaviors.[14,17(b),25] A similar approach has been applied recently to a simplified model with isotropic perturbations $\propto r^p$ of the Coulomb potential.[26] Many features of the equivalence can already be studied in this toy model. For the physically interesting case of the Zeeman Hamiltonian (1), the equivalent oscillator system is derived in Sec. II.

The main body of the paper focuses on exploiting this equivalence and is organized as follows. In Sec. III explicit transformation formulas are given which map perturbation expansions of both systems onto each other. The large-order behavior of perturbation expansions for the oscillator system is discussed in Sec. IV on the basis of the path-integral approach. Making use of known results for isotropic anharmonic oscillators, it turns out to be straightforward to derive a large-order formula for the ground-state energy which is equivalent to Eq. (2). This formula is checked against exact perturbation coefficients in relatively high order, and correction terms are computed numerically. The final two sections are devoted to a discussion of future applications and some concluding remarks. In Appendix A recursion relations are derived which generate the perturbation coefficients of the oscillator system, and in Appendix B an alternative approach is presented for calculating fluctuation determinants.

II. EQUIVALENCE

In this section we shall derive the equivalence of the Zeeman Hamiltonian (1) to a certain anisotropic anharmonic oscillator in four dimensions.

A. Heuristic considerations

In order to see the general aspects of the equivalence most transparently, we shall start with some heuristic considerations. They are based on the famous Kustaanheimo-Stiefel mapping[23]

$$\begin{aligned} x &= 2(x_1x_3 + x_2x_4)\,, \\ y &= 2(x_1x_4 - x_2x_3)\,, \\ z &= (x_1^2 + x_2^2) - (x_3^2 + x_4^2) \end{aligned} \tag{3}$$

between coordinates in three and four dimensions, satisfying

$$(x^2 + y^2 + z^2)^{1/2} \equiv r = \vec{\mathrm{x}}^2 \equiv x_1^2 + x_2^2 + x_3^2 + x_4^2\,. \tag{4}$$

(Boldface letters are three-vectors, and super-arrowed letters denote four-vectors.) Originally, this mapping was invented to cope with the classical Kepler problem. More recently, it was exploited by Duru and Kleinert[24] to obtain the path-integral quantization of the pure Coulomb potential, which had eluded any direct treatments before. Compared with Duru and Kleinert,[24] we have interchanged $x_1 \leftrightarrow x_3$. Here we shall not go into the details of their pioneering work on path integration. Rather, we shall only extract the main result relevant for our purposes which can be stated as follows. The Schrödinger equation $[-\frac{1}{2}\Delta + V(r)]\psi = E\psi$ for the pure Coulomb potential $V(r) = -Z/r$ is equivalent to the equation

$$[-\tfrac{1}{2}\vec{\nabla}^2 + \vec{\mathrm{x}}^2(-E/4 - Z/r)]\phi = 0\,, \tag{5}$$

where $\vec{\nabla}^2$ is the four-dimensional Laplacian. Using (4) and defining $\omega^2 \equiv -E/2$, $\varepsilon \equiv Z$, this can be written as the Schrödinger equation for a four-dimensional harmonic oscillator

$$\left(-\frac{1}{2}\vec{\nabla}^2 + \frac{\omega^2}{2}\vec{\mathrm{x}}^2\right)\phi = \varepsilon\phi. \tag{6}$$

Since the oscillator energies ε are functions of ω, the Coulomb energies E are given as solutions of the equation

$$\varepsilon = \varepsilon(\omega) = \varepsilon(\sqrt{-E/2}) = Z. \tag{7}$$

In four dimensions, the oscillator energies are $\varepsilon^{(n)} = \omega(n+2)$, so that $E^{(m)} = -\frac{1}{2}Z^2/(n/2+1)^2 = -\frac{1}{2}Z^2/m^2$, with principal quantum number $m \equiv n/2 + 1$. Obviously, a more rigorous treatment must yield an additional "selection rule" which restricts the solutions of (6) to those with quantum numbers n = even or, equivalently, even angular momentum (indeed, the proper path-integral treatment in Ref. 24 does precisely that). Such constraints are necessary to reduce the dynamical degrees of freedom of the four-dimensional system. It is exactly this subtlety which we shall neglect for the moment in our heuristic considerations.

It is straightforward to generalize this equivalence to a Coulomb potential plus isotropic perturbation.[26] If the total potential is $V(r) = -Z/r + gr^p$, then it is obvious from Eqs. (4)–(6), apart from a normalization factor, that this maps onto

$$\left(-\frac{1}{2}\vec{\nabla}^2 + \frac{\omega^2}{2}\vec{\mathrm{x}}^2 + \frac{g}{4^{p+1}}\vec{\mathrm{x}}^2(\vec{\mathrm{x}}^2)^p\right)\phi = \varepsilon\phi\,, \tag{8}$$

with $\varepsilon = \varepsilon(\omega, g/4^{p+1}) = \varepsilon(\sqrt{-E/2}, g/4^{p+1}) = Z$. Exactly as in the pure Coulomb case, the selection rule turns out[26] to be $\ell_O = 2\ell_C$, where ℓ_O and ℓ_C are, respectively, the angular momentum quantum numbers of the oscillator and Coulomb system.

Similarly, for the Stark Hamiltonian with potential $V(x,y,z) = -Z/r + gz$, we obtain

$$\left(-\frac{1}{2}\vec{\nabla}^2\frac{\omega^2}{2}\vec{\mathrm{x}}^2 + \frac{g}{4^2}\vec{\mathrm{x}}^2[(x_1^2 + x_2^2) - (x_3^2 + x_4^2)]\right)\phi = \varepsilon\phi\,. \tag{9}$$

This separates into two decoupled two-dimensional anharmonic oscillators [$\vec{u} = (u_1, u_2), \vec{\partial}^2 = \partial_{u_1}^2 + \partial_{u_2}^2$]

$$\left(-\frac{1}{2}\vec{\partial}^2+\frac{\omega^2}{2}\vec{u}^2\pm\frac{g}{4^2}(\vec{u}^2)^2\right)\phi_\pm=\varepsilon_\pm\phi_\pm\ , \tag{10}$$

leading to the following relation between ε and E:

$$\varepsilon=\varepsilon_+ +\varepsilon_- =\varepsilon_+(\sqrt{-E/2},+g/16) +\varepsilon_-(\sqrt{-E/2},-g/16)=Z\ . \tag{11}$$

The stable oscillator with positive quartic coupling determines the real part of the Stark resonances and the unstable one their decay rates. Of course, this equivalence is well known[14,17(b),25] and was exploited in many studies of the Stark effect. In these works, however, it was derived along different lines by separating the original Coulomb problem in parabolic coordinates.

Finally, consider the main subject of this paper, namely the Zeeman Hamiltonian (1). Neglecting the term $BL_z/2$ for the moment (see below), the Zeeman potential simplifies to $V(x,y,z)=-Z/r+g(x^2+y^2)$ with $g\equiv B^2/8$. Using (3), a short calculation gives $x^2+y^2=4(x_1^2+x_2^2)(x_3^2+x_4^2)$, so that the equivalent four-dimensional system is seen to be an *anisotropic* anharmonic oscillator,

$$\left(-\frac{1}{2}\vec{\nabla}^2+\frac{\omega^2}{2}\vec{x}^2+\frac{g}{4^3}4\vec{x}^2(x_1^2+x_2^2)(x_3^2+x_4^2)\right)\phi=\varepsilon\phi\ , \tag{12}$$

again with the relation $\varepsilon(\sqrt{-E/2},g/4^3)=Z$ between the energies ε and E.

Of course, the main virtue of these heuristic considerations is the unified treatment of quite general Coulomb systems in three dimensions in terms of anharmonic oscillators in four dimensions.

B. Proof of equivalence

As mentioned above, we have neglected so far the somewhat subtle problem of additional "selection rules" on the four-dimensional system. Although such constraints can in principle be derived from the path-integral approach along the lines of Duru and Kleinert,[24] in the anisotropic case this becomes quite cumbersome. For the anisotropic anharmonic oscillator (12) we shall therefore infer the precise meaning of the equivalence from a more conventional derivation starting from the Schrödinger equation for the Zeeman system,

$$H\psi\equiv\left(-\frac{1}{2}\Delta-\frac{Z}{r}+g(x^2+y^2)+\frac{B}{2}L_z\right)\psi=E\psi\ , \tag{13}$$

with $g\equiv B^2/8$ and $L_z=-i(x\partial_y-y\partial_x)$. In cylindrical coordinates, $x=\rho\cos\alpha$, $y=\rho\sin\alpha$, $z=z$, this separates into

$$\left(-\frac{1}{2}(\partial_z^2+\partial_\rho^2)+\frac{\ell_C^2-1/4}{2\rho^2} -\frac{Z}{(z^2+\rho^2)^{1/2}}+g\rho^2+\frac{B}{2}\ell_C\right)\varphi=E\varphi\ , \tag{14}$$

with $\psi=\rho^{-1/2}\varphi e^{i\ell_C\alpha}$, and $\ell_C=0,\pm1,\ldots$ being the two-dimensional angular momentum ("magnetic") quantum number. Introducing parabolic coordinates ρ_1 and ρ_2 via

$$z=\rho_1^2-\rho_2^2,\qquad \rho=2\rho_1\rho_2, \tag{15}$$

Eq. (14) becomes

$$\left(-\frac{1}{2}\frac{1}{4(\rho_1^2+\rho_2^2)}(\partial_{\rho_1}^2+\partial_{\rho_2}^2)+\frac{1}{2}\frac{\ell_C^2-1/4}{4\rho_1^2\rho_2^2} -\frac{Z}{\rho_1^2+\rho_2^2}+g4\rho_1^2\rho_2^2\right)\varphi=\tilde{E}\varphi\ , \tag{16}$$

where

$$\tilde{E}\equiv E-B\ell_C/2\ . \tag{17}$$

Multiplying (16) by $4(\rho_1^2+\rho_2^2)$ and rearranging terms, we arrive at

$$\left(-\frac{1}{2}(\partial_{\rho_1}^2+\partial_{\rho_2}^2)+\frac{\ell_C^2-1/4}{2\rho_1^2}+\frac{\ell_C^2-1/4}{2\rho_2^2} +(-4\tilde{E})(\rho_1^2+\rho_2^2)+4g4(\rho_1^2+\rho_2^2)\rho_1^2\rho_2^2\right)\varphi=4Z\varphi\ , \tag{18}$$

where use was made of the trivial identity $(\rho_1^2+\rho_2^2)/\rho_1^2\rho_2^2=1/\rho_1^2+1/\rho_2^2$. The transformations (14)–(18) are of course analogous to the treatment of the Stark Hamiltonian in parabolic coordinates [as can be seen by replacing $g\rho^2\longrightarrow gz$ in (14), and noting that the term $\propto g$ in (18) then becomes $4g(\rho_1^2+\rho_2^2)(\rho_1^2-\rho_2^2)=4g(\rho_1^4-\rho_2^4)$]. The crucial observation is now that Eq. (18) can be interpreted as the Schrödinger equation of two coupled two-dimensional anharmonic oscillators in cylindrical polar coordinates subject to the constraint

$$\ell_O^{(1)}=\ell_O^{(2)}=\ell_C\ . \tag{19}$$

Hence going back to Cartesian coordinates ($x_1=\rho_1\cos\alpha_1$, $x_2=\rho_1\sin\alpha_1$, $x_3=\rho_2\cos\alpha_2$, $x_4=\rho_2\sin\alpha_2$), we recover after a further trivial rescaling exactly the heuristic result (12) [but now supplemented with the selection rule (19) and with the BL_z term taken into account]:

$$h\phi\equiv\left(-\frac{1}{2}\vec{\nabla}^2+\frac{\omega^2}{2}\vec{x}^2+\lambda4\vec{x}^2(x_1^2+x_2^2)(x_3^2+x_4^2)\right)\phi =\varepsilon\phi\ , \tag{20}$$

with

$$\omega^2\equiv-\tilde{E}/2,\qquad \lambda\equiv g/4^3\ . \tag{21}$$

The energies ε and $\tilde{E}\equiv E-B\ell_C/2$ are related by the equation

$$\varepsilon(\omega,\lambda)=\varepsilon(\sqrt{-\tilde{E}/2},g/4^3)=Z. \tag{22}$$

In the following sections this relation will be used to deduce properties of the Zeeman system from those of the anisotropic anharmonic oscillator in (20). Notice that constraints like $\ell_O = 2\ell_C$ mentioned below, Eq. (8) yields, in general, the complete answer only as far as the energies are concerned. They are usually not strong enough to permit a unique relation between the wave functions also. In this case further constraints are necessary which reduce the degeneracy of the four-dimensional system. In our derivation of (20)–(22), on the other hand, the constraint (19) is already sufficient in the stronger latter sense. More precisely, for $g = 0$, the products of the two wave functions of the two-dimensional oscillators in (18) with radial and angular quantum numbers $n_r^{(1)}, n_r^{(2)}$ and $\ell_O^{(1)} = \ell_O^{(2)} = \ell_C$ are in a one-to-one correspondence with the hydrogen wave functions $\psi_{n_r^{(1)} n_r^{(2)} \ell_C}$ in parabolic coordinates (with energy $E_{n_r^{(1)} n_r^{(2)} \ell_C} = -\frac{1}{2}Z^2/m^2, m \equiv n_r^{(1)} + n_r^{(2)} + |\ell_C| + 1$, and degeneracy m^2). As usual, the perturbed wave functions for $g \neq 0$ may be labeled in the same way.

III. SOLUTION OF THE EQUATION RELATING ε AND E

In this section we analyze some consequences of the equivalence of the Hamiltonians (1) and (20). In particular we shall derive the large-order behavior of perturbation series expansions for Zeeman energies from that of the equivalent oscillator system.

As far as the scaling properties of the Hamiltonian h in (20) are concerned, there are no differences from a simplified model system with isotropic perturbation $\propto |\vec{x}|^6$. For the solution of (22) in terms of $\tilde{E}$ we can therefore apply most of the results derived in recent work[26] on general isotropic perturbations $\propto |\vec{x}|^{2(p+1)}$ [corresponding to perturbations $\propto r^p$ of the Coulomb potential, see (8)].

A. Exact energies

It is easy to show[26] that the energies $\varepsilon(\omega, \lambda)$ in (20) scale as

$$\varepsilon(\omega, \lambda) = \omega \hat{\varepsilon}(\hat{\lambda}), \quad \hat{\lambda} = \lambda/\omega^4 , \tag{23}$$

where $\hat{\varepsilon}$ and $\hat{\lambda}$ are reduced energy and coupling. Using (22) (with $Z \equiv 1$), this implies a parametric representation for $\tilde{E} = \tilde{E}(g)$,

$$\begin{aligned} \tilde{E} &= -2/\hat{\varepsilon}(\hat{\lambda})^2 , \\ g &= 4^3 \hat{\lambda}/\hat{\varepsilon}(\hat{\lambda})^4 , \end{aligned} \tag{24}$$

which might be a useful alternative for numerical computations of Zeeman energies.

B. Low-order perturbation theory

According to Eq. (23), perturbation expansions of the oscillator system must be of the form

$$\varepsilon = \omega \sum_{k=0} \varepsilon_k (\lambda/\omega^4)^k . \tag{25}$$

Inserting this expansion in Eq. (22) and solving it perturbatively, the coefficients E_k of the Zeeman series

$$\tilde{E} = \sum_{k=0} E_k g^k \tag{26}$$

can be expressed in terms of ε_l with $l \leq k$ as shown in Table I up to $k = 10$. (For technical details, see Ref. 26.)

Let us now apply the relations in Table I to the ground-state energies, to be denoted shortly by ε and E, omitting the superscript (0). Clearly, we then have $\varepsilon_0 = D/2 = 2$, and the first two coefficients ε_1 and ε_2 can easily be determined from standard perturbation theory as given in any textbook on quantum mechanics. Using obvious symmetry properties, we obtain

$$\varepsilon_1 = \langle 0000 | 4\vec{x}^2 (x_1^2 + x_2^2)(x_3^2 + x_4^2) | 0000 \rangle = 32[(x^4)_{00} + (x^2)_{00}^2](x^2)_{00} = 16 , \tag{27}$$

$$\begin{aligned} \varepsilon_2 = -4^2 \sum_{\vec{n}(\neq 0)} \{ & [(x^4)_{0n_1} \delta_{0n_2} + (x^4)_{0n_2} \delta_{0n_1} + 2(x^2)_{0n_1} (x^2)_{0n_2}] \\ & \times [(x^2)_{0n_3} \delta_{0n_4} + (x^2)_{0n_4} \delta_{0n_3}] + (n_1 \leftrightarrow n_3, n_2 \leftrightarrow n_4) \}^2 / (n_1 + n_2 + n_3 + n_4) \\ = -4^2 \tfrac{166}{3} , & \end{aligned} \tag{28}$$

with $(x^k)_{0n} \equiv \langle 0 | x^k | n \rangle$ denoting the well-known (one-dimensional) harmonic oscillator matrix elements. Their nonvanishing values are $(x^2)_{00} = \frac{1}{2}$, $(x^2)_{02} = \frac{1}{\sqrt{2}}$, $(x^4)_{00} = \frac{3}{4}$, $(x^4)_{02} = \frac{3}{2}\sqrt{2}$, $(x^4)_{04} = \frac{1}{2}\sqrt{6}$, and δ_{0n} is the usual Kronecker delta symbol. Inserting the numbers for ε_0, ε_1, and ε_2 in the expressions for E_k given in Table I, we recover the known Zeeman coefficients[7] up to second order:

$$\begin{aligned} E_0 &= -2/\varepsilon_0^2 = -\tfrac{1}{2} , \\ E_1 &= 4\varepsilon_0 \varepsilon_1 / 4^3 = 2 , \\ E_2 &= \varepsilon_0^4 (10\varepsilon_1^2 + 4\varepsilon_2 \varepsilon_0)/4^6 = -\tfrac{53}{3} . \end{aligned} \tag{29}$$

In higher orders, the general scheme of standard per-

turbation theory becomes very tedious. It is therefore preferable to calculate the ε_k by means of more specialized recursions, which are derived for the anisotropic oscillator system in Appendix A. Solving these recursions with the help of an algebraic computer program [written in REDUCE (Ref. 27)], it is quite easy to determine the low-order coefficients as exact, rational numbers. The first ten coefficients are given in Table II. When these numbers are inserted in Table I, we find for the ground-state energy of the Zeeman system the coefficients E_k listed in Table III. These numbers are in perfect agreement with results[7] derived directly from the Zeeman Hamiltonian (1).

C. Large-order perturbation theory

In order to relate ε_k and E_k in the large-order limit $k \to \infty$, it is convenient to apply the approach in Ref. 26 which takes advantage of the intimate relationship between perturbation coefficients in large order and the imaginary part of the energies for *negative* coupling constant.[28]

Let us briefly recall the main steps. For oscillators with perturbations of the type $\lambda x^{2(p+1)}$, the imaginary part of the energies for negative coupling are generically of the form[16–20]

$$\mathrm{Im}\varepsilon(\omega,\lambda) \underset{\lambda\to 0-}{\sim} \omega\gamma \left(\frac{1}{(a|\lambda|/\omega^{2+p})^{1/p}}\right)^{b} \times e^{-1/(a|\lambda|/\omega^{2+p})^{1/p}} [\,1 + c_1(a|\lambda|/\omega^{2+p})^{1/p} + c_2(a|\lambda|/\omega^{2+p})^{2/p} + \cdots], \quad (30)$$

with the parameters $a, b, \gamma, c_1, c_2, \ldots$, depending on the detailed form of the perturbation and the spatial dimen-

TABLE I. Solution of Eq. (22), defining the Zeeman perturbation coefficients E_k in terms of the perturbation coefficients $\varepsilon_l, l \le k$, of the anisotropic anharmonic oscillator in Eq. (20). To simplify the expressions, $\varepsilon_0 = 2$ has been inserted explicitly. The most general expressions for E_k, valid for any problem with $p = 2$, can be recovered by multiplying the right-hand side with a factor $(\varepsilon_0/2)^{4k-2}$ and replacing $\varepsilon_l \to 2(\varepsilon_l/\varepsilon_0)$.

$$
\begin{aligned}
E_0 &= -\tfrac{1}{2}\\
E_1 &= \varepsilon_1/8\\
E_2 &= (4\varepsilon_2 + 5\varepsilon_1^2)/128\\
E_3 &= (2\varepsilon_3 + 9\varepsilon_2\varepsilon_1 + 6\varepsilon_1^3)/256\\
E_4 &= (16\varepsilon_4 + 104\varepsilon_3\varepsilon_1 + 52\varepsilon_2^2 + 312\varepsilon_2\varepsilon_1^2 + 143\varepsilon_1^4)/8192\\
E_5 &= (4\varepsilon_5 + 34\varepsilon_4\varepsilon_1 + 34\varepsilon_3\varepsilon_2 + 136\varepsilon_3\varepsilon_1^2 + 136\varepsilon_2^2\varepsilon_1 + 340\varepsilon_2^3\varepsilon_1 + 119\varepsilon_1^5)/8192\\
E_6 &= (64\varepsilon_6 + 672\varepsilon_5\varepsilon_1 + 672\varepsilon_4\varepsilon_2 + 3360\varepsilon_4\varepsilon_1^2 + 336\varepsilon_3^2 + 6720\varepsilon_3\varepsilon_2\varepsilon_1\\
&\quad +10\,640\varepsilon_3\varepsilon_1^3 + 1120\varepsilon_2^3 + 15\,960\varepsilon_2^2\varepsilon_1^2 + 23\,940\varepsilon_2\varepsilon_1^4 + 6783\varepsilon_1^6)/524\,288\\
E_7 &= (16\varepsilon_7 + 200\varepsilon_6\varepsilon_1 + 200\varepsilon_5\varepsilon_2 + 1200\varepsilon_5\varepsilon_1^2 + 200\varepsilon_4\varepsilon_3 + 2400\varepsilon_4\varepsilon_2\varepsilon_1\\
&\quad +4600\varepsilon_4\varepsilon_1^3 + 1200\varepsilon_3^2\varepsilon_1 + 1200\varepsilon_3\varepsilon_2^2 + 13\,800\varepsilon_3\varepsilon_2\varepsilon_1^2 + 12\,650\varepsilon_3\varepsilon_1^4\\
&\quad +4600\varepsilon_2^3\varepsilon_1 + 25\,300\varepsilon_2^2\varepsilon_1^3 + 26\,565\varepsilon_2\varepsilon_1^5 + 6325\varepsilon_1^7)/524\,288\\
E_8 &= (256\varepsilon_8 + 3712\varepsilon_7\varepsilon_1 + 3712\varepsilon_6\varepsilon_2 + 25\,984\varepsilon_6\varepsilon_1^2 + 3712\varepsilon_5\varepsilon_3 + 51\,968\varepsilon_5\varepsilon_2\varepsilon_1\\
&\quad +116\,928\varepsilon_5\varepsilon_1^3 + 1856\varepsilon_4^2 + 51\,968\varepsilon_4\varepsilon_3\varepsilon_1 + 25\,984\varepsilon_4\varepsilon_2^2 + 350\,784\varepsilon_4\varepsilon_2\varepsilon_1^2\\
&\quad +380\,016\varepsilon_4\varepsilon_1^4 + 25\,984\varepsilon_3^2\varepsilon_2 + 175\,392\varepsilon_3^2\varepsilon_1^2 + 350\,784\varepsilon_3\varepsilon_2^2\varepsilon_1\\
&\quad +1\,520\,064\varepsilon_3\varepsilon_2\varepsilon_1^3 + 950\,040\varepsilon_3\varepsilon_1^5 + 29\,232\varepsilon_2^4 + 760\,032\varepsilon_2^3\varepsilon_1^2 + 2\,375\,100\varepsilon_2^2\varepsilon_1^4\\
&\quad +1\,900\,080\varepsilon_2\varepsilon_1^6 + 390195\varepsilon_1^8)/33\,554\,432\\
E_9 &= (64\varepsilon_9 + 1056\varepsilon_8\varepsilon_1 + 1056\varepsilon_7\varepsilon_2 + 8448\varepsilon_7\varepsilon_1^2 + 1056\varepsilon_6\varepsilon_3 + 16\,896\varepsilon_6\varepsilon_2\varepsilon_1\\
&\quad +43\,648\varepsilon_6\varepsilon_1^3 + 1056\varepsilon_5\varepsilon_4 + 16\,896\varepsilon_5\varepsilon_3\varepsilon_1 + 8448\varepsilon_5\varepsilon_2^2 + 130\,944\varepsilon_5\varepsilon_2\varepsilon_1^2\\
&\quad +163\,680\varepsilon_5\varepsilon_1^4 + 8448\varepsilon_4^2\varepsilon_1 + 16\,896\varepsilon_4\varepsilon_3\varepsilon_2 + 130\,944\varepsilon_4\varepsilon_3\varepsilon_1^2 + 130\,944\varepsilon_4\varepsilon_2^2\varepsilon_1\\
&\quad +654\,720\varepsilon_4\varepsilon_2\varepsilon_1^3 + 474\,672\varepsilon_4\varepsilon_1^5 + 2816\varepsilon_3^3 + 130\,944\varepsilon_3^2\varepsilon_2\varepsilon_1 + 327\,360\varepsilon_3^2\varepsilon_1^3\\
&\quad +43\,648\varepsilon_3\varepsilon_2^3 + 982\,080\varepsilon_3\varepsilon_2^2\varepsilon_1^2 + 2\,373\,360\varepsilon_3\varepsilon_2\varepsilon_1^4 + 1\,107\,568\varepsilon_3\varepsilon_1^6\\
&\quad +163\,680\varepsilon_2^4\varepsilon_1 + 1\,582\,240\varepsilon_2^3\varepsilon_1^3 + 3\,322\,704\varepsilon_2^2\varepsilon_1^5 + 2\,136\,024\varepsilon_2\varepsilon_1^7 + 385\,671\varepsilon_1^9)/33\,554\,432\\
E_{10} &= (256\varepsilon_{10} + 4736\varepsilon_9\varepsilon_1 + 4736\varepsilon_8\varepsilon_2 + 42\,624\varepsilon_8\varepsilon_1^2 + 4736\varepsilon_7\varepsilon_3 + 85\,248\varepsilon_7\varepsilon_2\varepsilon_1\\
&\quad +248\,640\varepsilon_7\varepsilon_1^3 + 4736\varepsilon_6\varepsilon_4 + 85\,248\varepsilon_6\varepsilon_3\varepsilon_1 + 42\,624\varepsilon_6\varepsilon_2^2 + 745\,920\varepsilon_6\varepsilon_2\varepsilon_1^2\\
&\quad +1\,056\,720\varepsilon_6\varepsilon_1^4 + 2368\varepsilon_5^2 + 85\,248\varepsilon_5\varepsilon_4\varepsilon_1 + 85\,248\varepsilon_5\varepsilon_3\varepsilon_2 + 745\,920\varepsilon_5\varepsilon_3\varepsilon_1^2\\
&\quad +745\,920\varepsilon_5\varepsilon_2^2\varepsilon_1 + 4\,226\,880\varepsilon_5\varepsilon_2\varepsilon_1^3 + 3\,487\,176\varepsilon_5\varepsilon_1^5 + 42\,624\varepsilon_4^2\varepsilon_2\\
&\quad +372\,960\varepsilon_4^2\varepsilon_1^2 + 42\,624\varepsilon_4\varepsilon_3^2 + 1\,491\,840\varepsilon_4\varepsilon_3\varepsilon_2\varepsilon_1 + 4\,226\,880\varepsilon_4\varepsilon_3\varepsilon_1^3\\
&\quad +248\,640\varepsilon_4\varepsilon_2^3 + 6\,340\,320\varepsilon_4\varepsilon_2^2\varepsilon_1^2 + 17\,435\,880\varepsilon_4\varepsilon_2\varepsilon_1^4 + 9\,299\,136\varepsilon_4\varepsilon_1^6\\
&\quad +248\,640\varepsilon_3^3\varepsilon_1 + 372\,960\varepsilon_3^2\varepsilon_2^2 + 6\,340\,320\varepsilon_3^2\varepsilon_2\varepsilon_1^2 + 8\,717\,940\varepsilon_3^2\varepsilon_1^4\\
&\quad +4\,226\,880\varepsilon_3\varepsilon_2^3\varepsilon_1 + 34\,871\,760\varepsilon_3\varepsilon_2^2\varepsilon_1^3 + 55\,794\,816\varepsilon_3\varepsilon_2\varepsilon_1^5 + 20\,590\,944\varepsilon_3\varepsilon_1^7\\
&\quad +211\,344\varepsilon_2^5 + 8\,717\,940\varepsilon_2^4\varepsilon_1^2 + 46\,495\,680\varepsilon_2^3\varepsilon_1^4 + 72\,068\,304\varepsilon_2^2\varepsilon_1^6\\
&\quad +38\,608\,020\varepsilon_2\varepsilon_1^8 + 6\,220\,181\varepsilon_1^{10})/536\,870\,912
\end{aligned}
$$

TABLE II. Perturbation coefficients for the ground-state energy of the anisotropic anharmonic oscillator in Eq. (20).

$\varepsilon_0 = 2$
$\varepsilon_1 = 4 \times 4^1$
$\varepsilon_2 = -166/3 \times 4^2$
$\varepsilon_3 = 18\,398/9 \times 4^3$
$\varepsilon_4 = -34\,571\,027/270 \times 4^4$
$\varepsilon_5 = 47\,713\,925\,953/4050 \times 4^5$
$\varepsilon_6 = -6\,702\,482\,253\,377/4500 \times 4^6$
$\varepsilon_7 = 7\,398\,933\,910\,500\,392\,213/29\,767\,500 \times 4^7$
$\varepsilon_8 = -1\,329\,676\,249\,040\,179\,764\,514\,127/25\,004\,700\,000 \times 4^8$
$\varepsilon_9 = 112\,178\,030\,738\,637\,487\,398\,233\,759\,053/7\,876\,480\,500\,000 \times 4^9$
$\varepsilon_{10} = -7\,739\,311\,896\,633\,212\,902\,498\,665\,902\,455\,303/1\,654\,060\,905\,000\,000 \times 4^{10}$

sion D. Since the parameter p determines "only" the scaling properties of the system, this includes isotropic as well as anisotropic perturbations. For the anisotropic anharmonic oscillator (20) (with $p = 2$ and $D = 4$), the parameters a, b, and γ will be calculated analytically in Sec. IV. The parameters $c_1, c_2, \ldots$, of the correction terms can then be estimated numerically by comparison with the exact perturbation coefficients ε_k.

The connection with the perturbation coefficients is given by the dispersion relation[16,17,29]

$$\omega\epsilon_k/(\omega^{2+p})^k = \frac{1}{\pi}\int_{-\infty}^{0} d\lambda \frac{\mathrm{Im}\varepsilon(\omega,\lambda+i0)}{\lambda^{k+1}}\,, \tag{31}$$

where the integral is taken along a cut in the complex coupling-constant plane. Inserting the small λ expansion (30), one finds the following large k behavior of the perturbation coefficients [recall Eq. (25)]

$$\varepsilon_k \underset{k\to\infty}{\sim} -\frac{p}{\pi}\gamma(-a)^k\Gamma(pk+b)\left(1+\frac{c_1}{pk+b-1}+\frac{c_2}{(pk+b-1)(pk+b-2)}+\cdots\right)\,. \tag{32}$$

This relationship suggests solving Eq. (22) first for small *negative* $\lambda = g/4^3$ and thus determining $\mathrm{Im}E$ in terms of $\mathrm{Im}\varepsilon$. The dispersion relation (31) gives then immediately the desired relation between ε_k and E_k as $k \to \infty$.

For negative coupling λ, Eq. (22) can be written as

$$1 = \varepsilon(\omega,\lambda) = \omega\sum_{k=0}\varepsilon_k\left(\frac{\lambda}{\omega^{2+p}}\right)^k + i\mathrm{Im}\varepsilon(\omega,\lambda)\,, \tag{33}$$

where we have added the real part of ε which, perturbatively, has the same expansion as for $\lambda > 0$. To solve this for $E = -2\omega^2$, we put $\omega = x + iy$ and note that $y \propto \mathrm{Im}\varepsilon$ is exponentially small. A straightforward perturbative solution[26] yields then

$$\mathrm{Im}E = -4xy = \frac{4}{\varepsilon_0^3}\gamma\left(\frac{1}{(\varepsilon_0^{2+p}a|\lambda|)^{1/p}}\right)^b \times e^{-[1/(\varepsilon_0^{2+p}a|\lambda|)^{1/p}](1+\delta_1)}(1+\delta_2)\,, \tag{34}$$

where

$$\delta_1 = (1+2/p)(\varepsilon_1/\varepsilon_0)\varepsilon_0^{2+p}|\lambda| + O(\lambda^2)\,, \tag{35}$$

$$\delta_2 = c_1(\varepsilon_0^{2+p}a|\lambda|)^{1/p} + [1-p+b(1+2/p)] \times(\varepsilon_1/\varepsilon_0)\varepsilon_0^{2+p}|\lambda| + O(|\lambda|^{1+1/p}) \tag{36}$$

TABLE III. Perturbation coefficients for the Zeeman ground-state energy calculated from the expressions in Table I with the ε_i from Table II.

$E_0 = -\frac{1}{2}$
$E_1 = 2$
$E_2 = -53/3$
$E_3 = 5581/9$
$E_4 = -21\,577\,397/540$
$E_5 = 31\,283\,298\,283/8100$
$E_6 = -13\,867\,513\,160\,861/27\,000$
$E_7 = 5\,337\,333\,446\,078\,164\,463/59\,535\,000$
$E_8 = -995\,860\,667\,291\,594\,211\,123\,017/50\,009\,400\,000$
$E_9 = 86\,629\,463\,423\,865\,975\,592\,742\,047\,423/15\,752\,961\,000\,000$
$E_{10} = -6\,127\,873\,544\,613\,551\,793\,091\,647\,103\,033\,033/3\,308\,121\,810\,000\,000$

are, for $p > 1$, subleading corrections. [Only for $p = 1$, does δ_1 contribute a factor $\exp(-\varepsilon_1/\varepsilon_0)$ to the leading behavior.[26]] Thus, recalling $\lambda = g/4^{p+1}$ and specializing to $p = 2$, we find for the leading behavior of the imaginary part of the Zeeman energies

$$\mathrm{Im}E(g) \underset{g\to 0-}{\sim} \gamma^* \left(\frac{1}{(a^*|g|)^{1/2}}\right)^b e^{-1/(a^*|g|)^{1/2}} \times[1 + c_1^*(a^*|g|)^{1/2} + c_2^*(a^*|g|) + \cdots] , \quad (37)$$

with

$$a^* = (\varepsilon_0^4/64)a \quad (38)$$

and

$$\gamma^* = (4/\varepsilon_0^3)\gamma \quad (39)$$

expressed in terms of the parameters ε_0, a, and γ of the anisotropic anharmonic oscillator (20), to be determined in Sec. IV. Furthermore, expanding in Eq. (34) the terms containing δ_i and using $\varepsilon_0^4 a|\lambda| = a^*|g|$, we obtain for $p = 2$

$$c_1^* = c_1 - \frac{2}{a}(\varepsilon_1/\varepsilon_0) , \quad (40)$$

$$c_2^* = c_2 - \frac{2}{a}(\varepsilon_1/\varepsilon_0)c_1 + \frac{2b-1}{a}(\varepsilon_1/\varepsilon_0) + \frac{2}{a^2}(\varepsilon_1/\varepsilon_0)^2 . \quad (41)$$

Finally, assuming the validity of a dispersion relation similar to (31) also for the Zeeman system, we see that the large-order behavior of E_k is given directly by that of ε_k in (32), if we replace a, γ, c_i by a^*, γ^*, c_i^*. In general, the relation between (30) and (37)–(39) holds for all equivalent oscillator and Coulomb systems with $p = 2$ (determining the degree of anharmonicity and henceforth the scaling properties).

IV. LARGE-ORDER ESTIMATE FOR THE ANISOTROPIC OSCILLATOR SYSTEM

Let us now turn to the calculation of the large-order parameters p, a, b, γ for the ground-state energy of the anisotropic anharmonic oscillator (20). Using Eqs. (37)–(39) this yields then quite directly a large-order formula for the Zeeman coefficients E_k. In order to display the symmetries of (20) more directly, it is useful to introduce the notation $\vec{x} = (\vec{u}, \vec{v}) = (u_1, u_2, v_1, v_2)$ so that Eq. (20) becomes

$$h\phi \equiv \left(-\frac{1}{2}(\vec{\partial}_u^2 + \vec{\partial}_v^2) + \frac{\omega^2}{2}(\vec{u}^2 + \vec{v}^2) + \lambda 4(\vec{u}^2 + \vec{v}^2)\vec{u}^2\vec{v}^2 \right)\phi = \varepsilon\phi . \quad (42)$$

Our calculation is based on the path-integral approach in Langer's formulation[30] which exploits the relationship between ε_k and $\mathrm{Im}\varepsilon(\lambda < 0)$ explained in Sec. III C. Starting point is the path-integral representation of the quantum partition function ($\beta = 1/k_B T$)

$$Z = \int \mathcal{D}^2 u \mathcal{D}^2 v\, e^{-\mathcal{A}[\vec{u},\vec{v}]} \underset{\beta\to\infty}{\longrightarrow} e^{-\beta\varepsilon} , \quad (43)$$

where

$$\mathcal{A}[\vec{u},\vec{v}] = \int_{-\beta/2}^{\beta/2} d\tau \left(\frac{1}{2}(\dot{\vec{u}}^2 + \dot{\vec{v}}^2) + \frac{\omega^2}{2}(\vec{u}^2 + \vec{v}^2) + 4\lambda(\vec{u}^2 + \vec{v}^2)\vec{u}^2\vec{v}^2 \right) \quad (44)$$

is the Euclidean action corresponding to the Hamiltonian in (42),

$$\int \mathcal{D}^2 u \mathcal{D}^2 v \equiv \prod_{n=1}^{N} \left(\int d^2 u_n (2\pi\Delta\tau)^{-1} \int d^2 v_n (2\pi\Delta\tau)^{-1} \right)$$

is, in the limit $N \to \infty$, the usual path-integral measure on a sliced "time" axis with spacing $\Delta\tau = \beta/N$, and the paths are assumed to be "periodic," i.e., $\vec{u}(-\beta/2) = \vec{u}(\beta/2)$, $\vec{v}(-\beta/2) = \vec{v}(\beta/2)$. Since the dependence on ω is known from the above scaling arguments, from now on we shall put $\omega \equiv 1$, for simplicity.

For $\lambda > 0$, the system is stable and Z is real. It may be computed, for instance, by perturbation theory, which, in the path-integral approach, corresponds to an expansion in fluctuations around the globally stable minimum at $(\vec{u}, \vec{v}) = 0$. For $\lambda < 0$, however, the system becomes unstable and Z, defined by analytic continuation, develops an imaginary part related to the decay rate. For small $\lambda < 0$, this rate is exponentially small and can be calculated by a saddle-point approximation (the analog of WKB). Rescaling variables by a factor $\lambda^{-1/4}$, it is easy to see that its dependence on λ must be of the form

$$\mathrm{Im}\, Z \propto e^{-\mathcal{A}_c} \equiv e^{-A/|\lambda|^{1/2}} \equiv e^{-1/(a|\lambda|)^{1/2}} , \quad (45)$$

where $A \equiv 1/a^{1/2}$ is the action (in the rescaled variables) of the saddle-point solution, to be referred to as "critical bubble" solution. The real part of Z follows from fluctuations around the locally stable minimum at $(\vec{u}, \vec{v}) = 0$. It has therefore the same power series expansion as for $\lambda > 0$,

$$\mathrm{Re}\, Z = e^{-(D/2)\beta[1+O(\lambda)]} . \quad (46)$$

Taking the large β limit and observing that, for small negative λ, $\mathrm{Im}Z$ is exponentially suppressed and hence much smaller than $\mathrm{Re}Z$, the ground-state resonance can be approximated by

$$\mathrm{Im}\,\varepsilon = -\frac{1}{\beta}\mathrm{Im}Z/\mathrm{Re}Z \propto e^{-1/(a|\lambda|)^{1/2}} ,$$

$$\mathrm{Re}\,\varepsilon = -\frac{1}{\beta}\ln \mathrm{Re}Z = \frac{D}{2}[1 + O(\lambda)] . \quad (47)$$

From the dispersion relation (31) we see that this simple scaling analysis determines already the leading large-order behavior $\varepsilon_k \approx (-a)^k \Gamma(2k)$, i.e., $p = 2$.

To determine the parameters b and γ, we must consider the fluctuations around the critical bubble. There is always one unstable fluctuation mode and a few so-called zero modes (Nambu-Goldstone modes) which leave the action invariant. The unstable mode produces the imaginary unit and is thus responsible for the decay of the system. The zero modes are associated with the symmetries of the system. Their multiplicity is directly related to the parameter b since it is well known[19,20] that each of them produces a factor

$$\propto (\mathcal{A}_c/2\pi)^{1/2} \propto (1/|\lambda|^{1/2})^{1/2} .$$

Being based on a path integral over periodic paths, one such mode is always associated with the freedom of translating the origin of the critical bubble solution along the τ axis. The other zero modes are related to internal symmetries like, e.g., rotational invariance. Consider first an isotropic oscillator in D dimensions. Its internal $O(D)$ symmetry gives rise to $D-1$ rotational zero modes, corresponding to the angular degrees of freedom of moving a vector over the surface of a D-dimensional sphere. Together with the translational zero mode this leads to a total prefactor

$$\propto (1/|\lambda|^{1/2})^{D/2} .$$

Our system (44) has $D = 4$, but due to the anisotropic perturbation its rotational symmetry is reduced to $O(2)\times O(2)$. Hence the sphere splits into two independent circles, so that there are only $1+1=2$ rotational zero modes, and the total prefactor becomes

$$\propto (1/|\lambda|^{1/2})^{3/2} .$$

This fixes the parameter $b = \frac{3}{2}$. Thus from scaling and symmetry arguments alone, we can conclude that

$$\varepsilon_k \propto (-a)^k \Gamma(2k+\tfrac{3}{2}), \tag{48}$$

$$E_k \propto (-a^*)^k \Gamma(2k+\tfrac{3}{2}). \tag{49}$$

Of course, in order to find the numerical value of a, we have to consider the critical bubble solution in more detail, and to determine γ, we must perform a careful fluctuation study which is the most laborious part. Fortunately, these calculations do not have to be done from scratch, since it turns out that many known results for isotropic anharmonic oscillators can be used.

A. The "critical bubble" solution

Looking at a contour plot of the potential in (42), $V(u,v) = \frac{1}{2}(u^2+v^2) + 4\lambda(u^2+v^2)u^2v^2$ with $u \equiv |\vec{u}|$, $v \equiv |\vec{v}|$, it is obvious that for $\lambda < 0$ the "tunneling path" with least action is along the diagonal ray $u = v \geq 0$. The directions of the two-dimensional vectors $\vec{u}$ and $\vec{v}$ are independent and can both be chosen arbitrarily, reflecting the $O(2)\times O(2)$ symmetry of the system. Any choice breaks this symmetry spontaneously and leads to rotational zero modes as described above. A convenient choice is

$$\vec{u} = \frac{1}{\sqrt{2}}(w,0) = \vec{v} , \tag{50}$$

since then the action reduces to that of an one-dimensional sextic anharmonic oscillator,

$$\mathcal{A} = \int d\tau (\tfrac{1}{2}\dot{w}^2 + \tfrac{1}{2}w^2 + \lambda w^6) , \tag{51}$$

with the usual normalizations. From studies of general self-interactions $\lambda w^{2(p+1)}$ it is known that this is extremized by the critical bubble solution (in the large β limit)[19,20]

$$\begin{aligned} w_c(\tau) &= \left(\frac{1}{2|\lambda|}\right)^{1/2p} \frac{1}{[\cosh p(\tau-\tau_0)]^{1/p}}\bigg|_{p=2} \\ &= \left(\frac{1}{2|\lambda|}\right)^{1/4} \frac{1}{[\cosh 2(\tau-\tau_0)]^{1/2}} , \end{aligned} \tag{52}$$

with associated action

$$\begin{aligned} \mathcal{A}_c &= 2^{1/p} \frac{\Gamma(1+1/p)^2}{\Gamma(2+2/p)} \frac{1}{|\lambda|^{1/p}}\bigg|_{p=2} \\ &\equiv A/|\lambda|^{1/2} \equiv 1/(a|\lambda|)^{1/2} . \end{aligned} \tag{53}$$

For $p = 2$, this gives $A = 2^{1/2}\pi/8$ and determines the parameter a to be

$$a = \left(\frac{1}{A}\right)^2 = \frac{32}{\pi^2}. \tag{54}$$

Also the origin τ_0 in (52) can be chosen arbitrarily. In the following paper we shall work with $\tau_0 = 0$, for simplicity. Any choice of τ_0 breaks spontaneously τ-translational invariance and gives rise to a translational zero mode. All these zero modes are encountered explicitly when dealing with fluctuations which we consider next.

B. Quadratic fluctuations

Expanding the full action (44) in the deviations $\delta\vec{u} \equiv \vec{u} - \vec{u}_c$, $\delta\vec{v} \equiv \vec{v} - \vec{v}_c$, we obtain up to quadratic order

$$\begin{aligned} \delta\mathcal{A} &\equiv \mathcal{A} - \mathcal{A}_c \\ &= \tfrac{1}{2}\int d\tau [\delta\dot{\vec{u}}^2 + \delta\dot{\vec{v}}^2 + \delta\vec{u}^2 + \delta\vec{v}^2 \\ &\quad + 8\lambda(\delta u_i \delta v_i)\underline{M}_{ij}(\delta u_j \delta v_j)^{\mathrm{t}} + \cdots] , \end{aligned} \tag{55}$$

where the superscript t denotes transposition and $\underline{M}_{ij}$ are the 2×2 matrices

$$\underline{M}_{ij} = \begin{pmatrix} [(\vec{v}_c^{\,2})^2 + 2\vec{u}_c^{\,2}\vec{v}_c^{\,2}]\delta_{ij} + 4(u_c)_i(u_c)_j\vec{v}_c^{\,2} & 4(u_c)_i(v_c)_j(\vec{u}_c^{\,2}+\vec{v}_c^{\,2}) \\ 4(u_c)_i(v_c)_j(\vec{u}_c^{\,2}+\vec{v}_c^{\,2}) & [(\vec{u}_c^{\,2})^2 + 2\vec{u}_c^{\,2}\vec{v}_c^{\,2}]\delta_{ij} + 4(v_c)_i(v_c)_j\vec{u}_c^{\,2} \end{pmatrix} . \tag{56}$$

In (55) summations over $i = 1, 2$, $j = 1, 2$ are implied. If the critical bubble solution (52), $(u_c)_1 = (v_c)_1 = w_c/\sqrt{2}$, $(u_c)_2 = (v_c)_2 = 0$, is inserted, $\underline{M}_{ij}$ decouples into a *longitudinal* and *transversal* part,

$$(\delta u_i \delta v_i)\underline{M}_{ij}(\delta u_j \delta v_j)^{\mathrm{t}} = (\delta u_1 \delta v_1)\underline{M}_{\mathrm{L}}(\delta u_1 \delta v_1)^{\mathrm{t}} + (\delta u_2 \delta v_2)\underline{M}_{\mathrm{T}}(\delta u_2 \delta v_2)^{\mathrm{t}} , \tag{57}$$

with

$$\underline{M}_{\mathrm{L}} = (w_c/\sqrt{2})^4 \begin{pmatrix} 7 & 8 \\ 8 & 7 \end{pmatrix} ,$$

$$\underline{M}_{\mathrm{T}} = (w_c/\sqrt{2})^4 \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} . \tag{58}$$

Since the quadratic terms decouple trivially, we can thus decompose (55) into $\delta\mathcal{A} = (\delta\mathcal{A})_{\mathrm{L}} + (\delta\mathcal{A})_{\mathrm{T}}$.

Consider first the *longitudinal* part. The eigenvalues of $\underline{M}_{\mathrm{L}}$ are $M_{\mathrm{L}}^{(1)} = 15(w_c/\sqrt{2})^4$ and $M_{\mathrm{L}}^{(2)} = -(w_c/\sqrt{2})^4$. The diagonalization of $\underline{M}_{\mathrm{L}}$ is achieved by a 45° rotation to new coordinates

$$\xi = \frac{1}{\sqrt{2}}(\delta u_1 + \delta v_1), \quad \eta = \frac{1}{\sqrt{2}}(\delta u_1 - \delta v_1). \tag{59}$$

Since the quadratic terms in (55) are invariant under such a rotation, we obtain for the longitudinal part of $\delta\mathcal{A}$

$$\begin{aligned}(\delta\mathcal{A})_{\mathrm{L}} &= \tfrac{1}{2}\int d\tau \left[\xi\left(-\frac{d^2}{d\tau^2} + 1 + 30\lambda w_c^4\right)\xi + \eta\left(-\frac{d^2}{d\tau^2} + 1 - 2\lambda w_c^4\right)\eta + \ldots\right] \\ &= \tfrac{1}{2}\int d\tau \left[\xi\left(-\frac{d^2}{d\tau^2} + 1 - \frac{15}{\cosh^2 2\tau}\right)\xi + \eta\left(-\frac{d^2}{d\tau^2} + 1 + \frac{1}{\cosh^2 2\tau}\right)\eta + \ldots\right] .\end{aligned} \tag{60}$$

The operator governing the ξ fluctuations coincides precisely with that of the one-dimensional x^6 oscillator. Its properties are therefore completely known.[19,20] In particular, it possesses an eigenmode ξ_- with *negative* eigenvalue $\kappa_- < 0$, indicating the expected metastability, and a zero mode $\xi_0 \propto \dot{w}_c$ with $\kappa_0 = 0$, associated with the translational invariance in τ, i.e., the freedom of choosing τ_0 in (52). Clearly, when the eigenmode expansion $\xi(\tau) = \sum_n \hat{\xi}_n \xi_n(\tau)$ is inserted in (60), the Gaussian integrals over these modes are formally divergent and require a careful treatment, which can be summarized by the following rules:[31]

$$\int d\hat{\xi}_- \frac{1}{\sqrt{2\pi}} e^{-\kappa_- \hat{\xi}_-^2/2} = (1/\kappa_-)^{1/2} \longrightarrow -\frac{i}{2}(1/|\kappa_-|)^{1/2} , \tag{61}$$

$$\int d\hat{\xi}_0 \frac{1}{\sqrt{2\pi}} e^{-\kappa_0 \hat{\xi}_0^2/2} = (1/\kappa_0)^{1/2} \longrightarrow \beta\left(\int d\tau \dot{w}_c^2/2\pi\right)^{1/2} . \tag{62}$$

The proper treatment of the decay mode ξ_- is based on the fact that, for $\lambda < 0$, Z must be defined by analytic continuation, and the second rule follows from the equality $\dot{w}_c d\tau_0 = \xi_0 d\hat{\xi}_0$. Making use of $\int d\tau \dot{w}_c^2 = \mathcal{A}_c$, $\kappa_- = -8$, and including the contribution of the Gaussian integrations over all other modes with positive eigenvalues, leads to the known fluctuation factor[19,20]

$$f_\xi = -\frac{i}{2}\left(\frac{1}{\pi 2^{-1/2} A}\right)^{1/2} \mathcal{A}_c^{1/2} \beta e^{-\beta/2} . \tag{63}$$

The operator associated with the η fluctuations has only positive eigenvalues. To see this we note that the potential term of this operator describes a barrier and not a well, so that it cannot sustain any bound state. In contrast to f_ξ, the fluctuation factor f_η is therefore simply a real number which, however, cannot be taken directly from the literature. We shall calculate it here by means of two alternative approaches. In the first approach, one starts with the trivial identity

$$f_\eta = \det\left(-\frac{d^2}{d\tau^2} + 1 + \frac{1}{\cosh^2 2\tau}\right)^{-1/2} \equiv f Z_{\mathrm{osc}} , \tag{64}$$

where

$$Z_{\mathrm{osc}} \equiv \det\left(-\frac{d^2}{d\tau^2} + 1\right)^{-1/2} = \frac{1}{2\sinh(\beta/2)} \underset{\beta\to\infty}{\sim} e^{-\beta/2} \tag{65}$$

is the partition function of the harmonic oscillator and

$$f \equiv \left(\frac{\det(-d^2/d\tau^2 + 1 + 1/\cosh^2 2\tau)}{\det(-d^2/d\tau^2 + 1)}\right)^{-1/2} . \tag{66}$$

After a change of variables, $t = 2\tau$, this ratio of determinants can be written in the canonical form

$$f \equiv \left(\frac{\det[-d^2/dt^2 + z - s(s+1)/\cosh^2 t]}{\det[-d^2/dt^2 + z]}\right)^{-1/2} , \tag{67}$$

with $z = \frac{1}{4}$ and $s = -\frac{1}{2}$. To this form a quite general formula[19,20] in the theory of Fredholm determinants is

applicable which states that

$$f = \left(\frac{\Gamma(\sqrt{z}-s)\Gamma(\sqrt{z}+1+s)}{\Gamma(\sqrt{z})\Gamma(\sqrt{z}+1)}\right)^{1/2} = \left(\frac{\Gamma(\frac{1}{2}+\frac{1}{2})\Gamma(\frac{1}{2}+1-\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+1)}\right)^{1/2} = (2/\pi)^{1/2}\,, \tag{68}$$

where $\Gamma()$ is the usual gamma function. In more physical terms, this formula can be derived by relating f to the quantum-mechanical transmission amplitude of one-dimensional scattering at the potential $z - s(s+1)/\cosh^2 t$. See the last entry of Ref. 20. Inserting (68) in (64) we obtain, in the large β limit,

$$f_\eta = (2/\pi)^{1/2} e^{-\beta/2}. \tag{69}$$

In Appendix B, this result is verified by a completely different approach based on a modified Gelfand-Yaglom formula.[32,33]

Let us now turn to the *transversal* fluctuations in (55)–(58). Since the matrix $\underline{M}_{\rm T}$ is proportional to unity, the transversal degrees of freedom of $\delta\vec{u}$ and $\delta\vec{v}$ decouple automatically. This leads to two identical fluctuation contributions to $\delta\mathcal{A}$,

$$(\delta\mathcal{A})_{\rm T} = \tfrac{1}{2}\int d\tau \left[\delta u_2 \left(-\frac{d^2}{d\tau^2}+1+6\lambda w_c^4\right)\delta u_2 + (\delta u_2 \leftrightarrow \delta v_2) + \cdots\right] = \tfrac{1}{2}\int d\tau \left[\delta u_2 \left(-\frac{d^2}{d\tau^2}+1-\frac{3}{\cosh^2 2\tau}\right)\delta u_2 + (\delta u_2 \leftrightarrow \delta v_2) + \cdots\right]. \tag{70}$$

Each of them is governed by the transversal operator of a general O(n) symmetric $|\vec{x}|^6$ oscillator whose properties are again known. It contains one zero mode $\propto w_c$, associated with the freedom of choosing the directions of $\vec{u}_c$ and $\vec{v}_c$, respectively. If the analog of (62) for rotational zero modes[19,20] (S_n is the surface of the n-dimensional unit sphere),

$$(1/\kappa_0)^{(n-1)/2} \longrightarrow S_n \left(\int d\tau\, \vec{u}_c^{\,2}/2\pi\right)^{(n-1)/2}, \tag{71}$$

is adapted to our case $n = 2$, we obtain

$$(1/\kappa_0)^{1/2} \longrightarrow 2\pi \left(\int d\tau\, \vec{u}_c^{\,2}/2\pi\right)^{1/2}, \tag{72}$$

with the prefactor 2π being the "surface" of the unit circle. Recalling that $\vec{u}_c^{\,2} = \frac{1}{2}w_c^2$ and using the scaling properties of the action (44), the integral in (72) can be expressed as

$$\left(\int d\tau\, \vec{u}_c^{\,2}/2\pi\right)^{1/2} = \frac{1}{\sqrt{2}}\left(\int d\tau\, w_c^2/2\pi\right)^{1/2} = \frac{1}{\sqrt{2}}(\mathcal{A}_c/\pi)^{1/2}\,. \tag{73}$$

Notice that there is an additional factor $1/\sqrt{2}$ compared with the ordinary O(2)-symmetric $|\vec{x}|^6$ oscillator.[19,20] Together with the known contribution[19,20] of all other modes with positive eigenvalues, we thus have

$$f_{\delta u_2} = f_{\delta v_2} = \frac{1}{\sqrt{2}}\left(\frac{1}{\pi 2^{-1/2}A}\right)^{1/2} \mathcal{A}_c^{1/2} 2\pi e^{-\beta/2}\,. \tag{74}$$

C. Large-order formula

Combining (63), (69), and (74) we obtain the final result

$$\begin{aligned}{\rm Im}Z &= -|f_\xi| f_\eta f_{\delta u_2} f_{\delta v_2} e^{-\mathcal{A}_c} \\ &= -\frac{1}{2}\left(\frac{2}{\pi}\right)^{1/2}\frac{(2\pi)^2}{2}\left(\frac{1}{\pi 2^{-1/2}A}\right)^{3/2} \times \mathcal{A}_c^{3/2} e^{-\mathcal{A}_c}\beta e^{-2\beta} \\ &= -\sqrt{2}\left(\frac{1}{2^{-1/2}A}\right)^{3/2}\mathcal{A}_c^{3/2} e^{-\mathcal{A}_c}\beta e^{-2\beta}\,,\end{aligned} \tag{75}$$

and consequently, using (46) and (47),

$$\begin{aligned}{\rm Im}\varepsilon &= \sqrt{2}\left(\frac{1}{2^{-1/2}A}\right)^{3/2}\mathcal{A}_c^{3/2} e^{-\mathcal{A}_c} \\ &= \sqrt{2}\left(\frac{8}{\pi}\right)^{3/2}\mathcal{A}_c^{3/2} e^{-\mathcal{A}_c} \\ &= \frac{32}{\pi^{3/2}}\mathcal{A}_c^{3/2} e^{-\mathcal{A}_c}\,,\end{aligned} \tag{76}$$

with $\mathcal{A}_c = 1/(a|\lambda|)^{1/2} = 1/(32|\lambda|/\pi^2)^{1/2}$ given in (53) and (54). Comparing with the general expression (30), we see that the large-order parameters p, a, b, γ are determined as

$$p = 2, \quad a = 32/\pi^2, \quad b = \tfrac{3}{2}, \quad \gamma = 32/\pi^{3/2}. \tag{77}$$

Hence, recalling the dispersion relation (31), we find for the ground-state expansion of the anisotropic anharmonic oscillator (20) the large-order behavior

$$\begin{aligned}\varepsilon_k &\underset{k\to\infty}{\sim} \varepsilon_k^{\rm asy} \equiv -\frac{p}{\pi}\gamma(-a)^k\Gamma(pk+b) \\ &= -\frac{64}{\pi^{5/2}}(-32/\pi^2)^k\Gamma(2k+\tfrac{3}{2})\,.\end{aligned} \tag{78}$$

The superscript "asy" stands to indicate that this is the leading asymptotic behavior for large k, omitting the $1/k$ corrections to be discussed below. Finally, using the relations (37)–(39) with $\varepsilon_0 = 2$ so that $a^* = a/4, \gamma^* = \gamma/2$, this implies for the ground-state energy of the Zeeman Hamiltonian the large-order formula

$$\begin{aligned}E_k &\underset{k\to\infty}{\sim} E_k^{\rm asy} \equiv -\frac{p}{\pi}\gamma^*(-a^*)^k\Gamma(pk+b) \\ &= -\frac{32}{\pi^{5/2}}(-8/\pi^2)^k\Gamma(2k+\tfrac{3}{2}),\end{aligned} \tag{79}$$

in agreement with the result quoted in (2).

D. Comparison with numerical results

Let us now test the accuracy of the large-order formula (78) for the anisotropic anharmonic oscillator by comparing it with exact coefficients in relatively high order (up to $k = 110$). The exact coefficients can be generated quite easily with the help of the recursions derived in Appendix A. As a side result, we shall get numerical estimates for the corrections c_i in [recall (32), and $p = 2, b = \frac{3}{2}$]

$$\varepsilon_k = \varepsilon_k^{\text{asy}} \left(1 + \frac{c_1}{2k+\frac{1}{2}} + \frac{c_2}{(2k+\frac{1}{2})(2k-\frac{1}{2})} + \cdots\right) , \tag{80}$$

which, by means of (40) and (41), can then be translated into the corresponding corrections to the Zeeman formula (79).

In Fig. 1(a), the ratio

$$R_k \equiv \varepsilon_k/\varepsilon_k^{\text{asy}} \tag{81}$$

is plotted versus k. We see that the expected limiting value $R_\infty = 1$ is reached very rapidly, indicating a relatively small correction term c_1. For numerical purposes it is convenient to reexpand the term in large parentheses in (80) to

$$\left(1 + \frac{\gamma_1}{k} + \frac{\gamma_2}{k^2} + \cdots\right) , \tag{82}$$

with

$$\begin{aligned} \gamma_1 &= c_1/2 , \\ \gamma_2 &= c_2/4 - c_1/8 . \end{aligned} \tag{83}$$

The values of these parameters can be estimated in Fig. 1(b), where the data in Fig. 1(a) are replotted in the form $(1 - R_k)k = -\gamma_1 - \gamma_2/k + \cdots$ versus $1/k$. A linear extrapolation to $1/k = 0$ yields $\gamma_1 \approx -0.15$, and from the asymptotic slope we read off $\gamma_2 \approx -4$. Notice that γ_1 is comparatively small. This is presumably related to the numerical observation[34] that for the *isotropic* oscillator with $|\vec{x}|^6$ perturbation the $1/k$ correction seems to vanish completely. (For $D = 1, \ldots, 4$ the accuracy of this result is better than 10^{-12}.) More precise values can be obtained from numerical Neville-like extrapolations,[35] to be compiled in Table IV. Formula (78) corrected by (82) with γ_1, γ_2 from Table IV is then accurate to within 0.16% for $k = 11$ and 0.000 003 6% for $k = 71$. The values derived for γ_i^* are consistent with direct fits to the Zeeman series which gave[7] $\gamma_1^* = -2.6183$ and $\gamma_2^* = 1.283$. On the other hand, Avron's[9] guess of an analytical expression, $\gamma_1^* = -3\pi^2/2^{7/2} \approx -2.617\,074$, deviates already in the fourth digit and is probably incorrect. While this analytical formula for γ_1^* looks quite "natural" for the Zeeman system, it leads to an "unnatural," complicated expression for the correction γ_1 of the equivalent oscillator system.

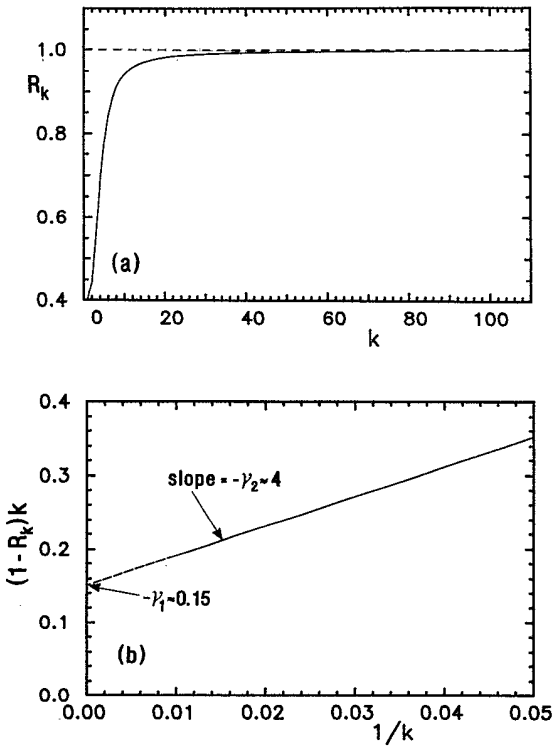

FIG. 1. (a) The ratio $R_k \equiv \varepsilon_k/\varepsilon_k^{\text{asy}} = 1+\gamma_1/k+\gamma_2/k^2+\cdots$ for the anisotropic anharmonic oscillator in Eq. (20). (b) The same data as in (a) but replotted in the form $(1 - R_k)k = -\gamma_1 - \gamma_2/k$ vs $1/k$. For more precise values of γ_1, γ_2 see Table IV.

TABLE IV. Higher-order corrections to the large-order formulas for the anisotropic anharmonic oscillator (c_i, γ_i) and the Zeeman ground-state energies (c_i^*, γ_i^*). The c's and γ's are related by $c_1 = 2\gamma_1$, $c_2 = 4\gamma_2 + \gamma_1$ [see Eq. (83)], and for c_i^* we have from Eqs. (40) and (41) (with $a = 32/\pi^2$, $b = \frac{3}{2}$ and $\varepsilon_1/\varepsilon_0 = \frac{16}{2} = 8$), $c_1^* = c_1 - \pi^2/2$, and $c_2^* = c_2 - (\pi^2/2)c_1 + \pi^2/2 + \pi^4/8$.

	Anharmonic oscillator		Zeeman	
i	γ_i	c_i	c_i^*	γ_i^*
1	−0.150 899 9	−0.301 799 8	−5.236 602 0	−2.618 301 0
2	−3.982 15	−16.079 50	2.520 76	1.284 77

V. DISCUSSION

Our calculation has demonstrated that the leading large-order behavior of Zeeman perturbation series expansions can be understood by simple scaling and symmetry arguments for an equivalent four-dimensional oscillator system. The detailed form has been derived by means of the path-integral approach applied to this oscillator system. It is therefore interesting to mention the doubts in Ref. 6 (see also Ref. 12) concerning the validity of the path-integral approach in Lipatov's formulation[36] when applied directly to the Zeeman problem. Although the present calculation is based on Langer's formulation (which is closely related to but not identical with Lipatov's), it is hardly conceivable that Lipatov's approach should break down when applied to the anisotropic anharmonic oscillator (20). Thus, albeit in a somewhat indirect fashion, the equivalent formulation opens the way to a path-integral treatment of large-order estimates for the Zeeman system.

From a more practical point of view it is important to notice that the perturbation coefficients for the oscillator system approach their asymptotic limit much more rapidly than the corresponding Zeeman coefficients. This might be quite advantageous in applications of resummation techniques like, e.g., Borel's method,[20] which make use of the large-order information. In the interesting case of strong magnetic fields it is presumably even more important that the oscillator energies show an algebraic strong-coupling behavior ($\epsilon \propto \lambda^{1/4}$ as $\lambda \to \infty$). In standard procedures, such a behavior can easily be taken into account to improve the convergence of resummation, as is known from the experience with isotropic (or one-dimensional) anharmonic oscillators. A direct resummation of the Zeeman series, on the other hand, is complicated by its nonalgebraic strong-coupling behavior containing terms of the type $\log g, \log(\log g), \ldots$ as $g \to \infty$,[11] and requires a special treatment.[15] In order to calculate precise Zeeman energies from perturbation theory this suggests that we resum first the well-behaved oscillator series and then map the resulting energies back onto the Zeeman system by means of the exact relations (24).

VI. CONCLUSION

Extending similar relationships for the Coulomb potential plus isotropic perturbations and the Stark Hamiltonian, the Zeeman Hamiltonian in *three* dimensions has been shown to be equivalent to an anisotropic anharmonic oscillator in *four* dimensions. Apart from the theoretical interest in unifying two seemingly different systems, this equivalence appears to be useful also from a practical point of view. This was demonstrated explicitly by showing that the large-order estimate for the ground-state expansion of the Zeeman Hamiltonian can be rederived quite easily from the equivalent anisotropic anharmonic oscillator system by using well-known Bender-Wu formulas for isotropic anharmonic oscillators. As a future application it would be interesting to investigate the accuracy of Zeeman energies calculated by first resumming the well-behaved oscillator series and then using the exact transformation formulas. Of course, other information available for anharmonic oscillator systems (such as rigorous inequalities) will also be transferable to the Zeeman system and may yield interesting new results.

Note added in proof. After the submission of this paper, Professor M. Kibler kindly pointed out to me that he and T. Negadi have emphasized the equivalence of the Zeeman Hamiltonian to an anisotropic anharmonic oscillator before in Lett. Nuovo Cimento **39**, 319 (1984). These authors did not, however, derive any consequences from their observation, as apparently intended. The main point of the present paper was to exploit this equivalence and deduce the large-order behavior of Zeeman perturbation series from path-integral analyses of the oscillator system.

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APPENDIX A: RECURSIONS FOR GROUND-STATE PERTURBATION COEFFICIENTS OF OSCILLATOR SYSTEM

In this appendix, recursions are derived for the perturbation coefficients ε_k of the ground-state energy $\varepsilon = 2 + \sum_{k=1} \varepsilon_k \lambda^k$ of the anisotropic oscillator system

$$[-\tfrac{1}{2}(\vec{\partial}_u^2 + \vec{\partial}_v^2) + \tfrac{1}{2}(\vec{u}^2 + \vec{v}^2) + 4\lambda(\vec{u}^2 + \vec{v}^2)\vec{u}^2\vec{v}^2]\psi = \varepsilon\psi \, . \tag{A1}$$

Such recursions are the basis for an efficient computation of ε_k in relatively high order (say, up to $k \approx 100$) of perturbation theory. Following a scheme introduced by Bender and Wu,[16] we proceed as follows.

Parametrizing $\vec{u}$ and $\vec{v}$ by two-dimensional polar coordinates, the radial ground-state Schrödinger equation becomes

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) - \frac{1}{2u}\frac{\partial}{\partial u} - \frac{1}{2v}\frac{\partial}{\partial v} + \frac{1}{2}(u^2 + v^2) + 4\lambda(u^2 + v^2)u^2v^2\right]\hat{\psi} = \varepsilon\hat{\psi} \, . \tag{A2}$$

Separating out the unperturbed ground-state wave function, $\psi_0 = e^{-u^2/2}e^{-v^2/2}$,

$$\hat{\psi} = \psi_0\phi = e^{-(u^2+v^2)/2}\phi \, , \tag{A3}$$

Eq. (A2) transforms into

$$\left[-\frac{1}{2}uv\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)-v\left(\frac{1}{2}-u^2\right)\frac{\partial}{\partial u}-u\left(\frac{1}{2}-v^2\right)\frac{\partial}{\partial v}+4\lambda(u^2+v^2)u^3v^3\right]\phi=uv(\varepsilon-2)\phi\,. \quad \text{(A4)}$$

Inserting the perturbation expansions [$b_0(u,v) \equiv 1 \equiv b_{0,0,0}$]

$$\phi(u,v)=\sum_{k=0} b_k(u,v)\lambda^k, \quad \varepsilon=2+\sum_{k=1}\varepsilon_k\lambda^k\,, \quad \text{(A5)}$$

and comparing equal powers in λ, we find

$$-\frac{1}{2}uv\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)b_k-v\left(\frac{1}{2}-u^2\right)\frac{\partial}{\partial u}b_k-u\left(\frac{1}{2}-v^2\right)\frac{\partial}{\partial v}b_k+4(u^2+v^2)u^3v^3b_{k-1}=uv\sum_{\ell=0}^{k-1}\varepsilon_{k-\ell}b_\ell\,. \quad \text{(A6)}$$

This can be solved with the ansatz

$$b_k(u,v)=\sum_{i,j=0}^{2k} b_{k,i,j}(u^2)^i(v^2)^j\,, \quad \text{(A7)}$$

leading to the recursion

$$b_{k,i,j}=\frac{1}{2(i+j)}\left(2(i+1)^2b_{k,i+1,j}+2(j+1)^2b_{k,i,j+1}-4b_{k-1,i-2,j-1}-4b_{k-1,i-1,j-2}+\sum_{\ell=1}^{k-1}\varepsilon_{k-\ell}b_{\ell,i,j}\right)\,, \quad \text{(A8)}$$

with

$$\varepsilon_k=-2(b_{k,1,0}+b_{k,0,1})\,. \quad \text{(A9)}$$

The recursion must be initialized with $b_{0,0,0}=1$ and all other $b_{k,i,j}=0$, and then worked through with increasing $k=1,2,3,\ldots$, and, for each k, decreasing $i=2k,\ldots,0$, $j=2k,\ldots,0$ (omitting $i=j=0$).

This recursion was run with an algebraic computer program written in REDUCE (Ref. 27) to generate the exact coefficients ε_k displayed in Table I. With a numerical program it was possible to compute ε_k up to order $k=110$ with ≈ 26 digits accuracy. Notice finally that the recursion yields, besides the energy expansion, also the wave-function expansion in a simple way.

APPENDIX B: FLUCTUATION DETERMINANT f_η FROM MODIFIED GELFAND-YAGLOM APPROACH

This appendix is devoted to an alternative derivation of formula (69) for the fluctuation determinant f_η defined in Eq. (64). It employs a modification of a formula by Gelfand and Yaglom[32] for fluctuation determinants with *fixed* boundary conditions [$\eta(-\beta/2)=\eta(\beta/2)=0$]. If these are replaced by *periodic* boundary conditions, the modified formula can be stated as follows:[33]

Let $D(\tau)$ be a solution of the differential equation

$$\mathcal{O}_\eta D(\tau)\equiv\left(-\frac{d^2}{d\tau^2}+1+\frac{1}{\cosh^2 2\tau}\right)D(\tau)=0\,, \quad \text{(B1)}$$

with initial conditions

$$D(-\beta/2)=0, \quad \dot{D}(-\beta/2)=1\,. \quad \text{(B2)}$$

Then

$$f_\eta=[\det\mathcal{O}_\eta]^{-1/2}=\left(\frac{1}{2[\dot{D}(\beta/2)-1]}\right)^{1/2}\,. \quad \text{(B3)}$$

The even solution of (B1) is

$$D_e(\tau)=\cosh^{1/2}(2\tau)\,, \quad \text{(B4)}$$

and the independent odd one follows from

$$D_o(\tau)=2D_e(\tau)\int_0^\tau\frac{d\tau'}{D_e^2(\tau')}=\cosh^{1/2}(2\tau)\arcsin(\tanh 2\tau)\,. \quad \text{(B5)}$$

The general solution

$$D(\tau)=AD_e(\tau)+BD_o(\tau) \quad \text{(B6)}$$

satisfies the initial conditions (B2) if we choose

$$A=\tfrac{1}{2}\cosh^{1/2}(\beta)\arcsin(\tanh\beta), \quad B=\tfrac{1}{2}\cosh^{1/2}(\beta). \quad \text{(B7)}$$

This gives

$$\dot{D}(\beta/2)=A\frac{\sinh\beta}{\cosh^{1/2}(\beta)}+B\left(\frac{\sinh\beta}{\cosh^{1/2}(\beta)}\arcsin(\tanh\beta)+\frac{2}{\cosh^{1/2}(\beta)}\right)=\sinh\beta\arcsin(\tanh\beta)+1\,, \quad \text{(B8)}$$

and inserting in (B3) we have

$$f_\eta=\left(\frac{1}{2\sinh\beta\arcsin(\tanh\beta)}\right)^{1/2}=\left(\frac{\tanh(\beta/2)}{\arcsin(\tanh\beta)}\right)^{1/2}Z_{\rm osc}\,, \quad \text{(B9)}$$

with $Z_{\rm osc}=1/[2\sinh(\beta/2)]$. Hence, in the limit $\beta\longrightarrow\infty$, we obtain once again Eq. (69),

$$f_\eta\underset{\beta\to\infty}{\sim}\sqrt{1/\arcsin(1)}e^{-\beta/2}=\sqrt{2/\pi}e^{-\beta/2}\,. \quad \text{(B10)}$$

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