ZEEMAN EFFECT FROM FOUR-DIMENSIONAL ANISOTROPIC ANHARMONIC OSCILLATOR

W. JANKE

Institut für Theorie der Elementarteilchen, Freie Universität Berlin, Arnimallee 14, D-1000 Berlin 33, Germany

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The Zeeman Hamiltonian for (spinless) hydrogen in constant magnetic field is shown to be equivalent to a four-dimensional anisotropic anharmonic oscillator. This relation is used to rederive the large-order behaviour of Rayleigh–Schrödinger perturbation series expansions for Zeeman energies in terms of known Bender–Wu formulae for anharmonic oscillators.

Ever since the work of Kustaanheimo and Stiefel [1] on the classical Kepler problem it has been known that three-dimensional Coulomb systems can be related to four-dimensional oscillators. More recently this relation was exploited by Duru and Kleinert [2] to obtain the path-integral quantization of the pure Coulomb potential in terms of the harmonic oscillator. For more general Coulomb systems, the equivalent oscillator will contain anharmonic terms and may in general be anisotropic.

A well-known example for such a generalization is the Stark Hamiltonian for a hydrogen atom in constant electric field. This system can be shown [3] to be equivalent to two decoupled two-dimensional isotropic oscillators with quartic anharmonicity. Employing this relation, Rayleigh–Schrödinger perturbation series expansions for Stark resonances were calculated [4] from those for oscillator energies. A particularly useful application is the derivation of estimates for the large-order behaviour of these Stark series [3–5] which, by making use of the corresponding known Bender–Wu formulae [6–8] for anharmonic oscillators, turned out to be straightforward in this approach.

For the Zeeman effect [9], on the other hand, observed when a hydrogen atom moves in a constant magnetic field, a similar approach has never been taken since the equivalent oscillator system was apparently not known. Consequently, perturbation series [10] and their large-order behaviours [11,5] were derived independently, staying completely within the Coulomb formulation. The purpose of this note is to close the gap by showing that the Zeeman effect can also be described in terms of a four-dimensional oscillator, with an anharmonic term which turns out to be of anisotropic, sextic type. Many features of this relation can already be investigated [12] in a simplified model with isotropic perturbations of the Coulomb potential. Here we shall focus on physical applications and demonstrate its usefulness by rederiving perturbation expansions for the Zeeman effect and their large-order properties from known results for the equivalent oscillator system.

In atomic units, the Zeeman Hamiltonian for spinless hydrogen (with infinitely heavy nucleus) in constant magnetic field $\boldsymbol{B}$ is given by

$$H = H_C + H_I = (\tfrac{1}{2}\boldsymbol{p}^2 - 1/r) + [g(x^2+y^2) + \tfrac{1}{2}BL_z] , \qquad (1)$$

where $\boldsymbol{p} = -\mathrm{i}\nabla$, $g \equiv B^2/8$, and L_z is the angular momentum operator in z direction, which is taken along the magnetic field direction, $\boldsymbol{B} = B\boldsymbol{e}_z$.

In atomic units, $B = 1$ corresponds to 2.35×10^9 G. Although this is extremely large compared with typical laboratory fields around 10^4 G, such strong magnetic fields are needed in astrophysics [13,14] to describe neutron star surfaces (10^{10}–10^{13} G) and white dwarf stars (10^7–10^8 G) as well as in some solid-state systems [14] to take into account *effective* interac-

tions (up to 10^{10} G). Many numerical and analytical investigations of this Hamiltonian have been reported in the literature [15]. For strong fields, usual low-order perturbation expansions are certainly not sufficient. Carrying them to higher orders seems, at first sight, useless since they are known to be asymptotic series with coefficients growing like $(2k)!$ [11]. Still, if such high-order perturbation expansions are combined with special resummation algorithms [16], they are a useful tool for calculating precise Zeeman energies, even for strong fields. For most of such algorithms, a necessary prerequisite is the detailed knowledge of the large-order behaviour of the perturbation expansions. For the Zeeman Hamiltonian this has been derived [11] on the basis of quite involved multidimensional WKB techniques. By means of the new equivalence we shall be able to reproduce these results very easily by using known Bender–Wu formulae [6–8] for anharmonic oscillators, similarly to the treatment of the Stark effect.

In order to motivate the new equivalence, let us start with a few heuristic considerations based directly on the Kustaanheimo–Stiefel mapping [1,2] from three to four dimensions,

$$x=2(x_1x_3+x_2x_4)\,,$$
$$y=-2(x_1x_4-x_2x_3)\,,$$
$$z=-(x_1^2+x_2^2)+(x_3^2+x_4^2)\,, \qquad (2)$$

satisfying

$$(x^2+y^2+z^2)^{1/2}\equiv r=\boldsymbol{x}^2\equiv x_1^2+x_2^2+x_3^2+x_4^2\,.$$

For the pure Coulomb term in (1), $V_C=-Z/r$, this mapping has been used in ref. [2] to show that the Schrödinger equation $H_C\psi=E_C\psi$ is equivalent to that of a four-dimensional harmonic oscillator,

$$(-\tfrac{1}{2}\nabla^2+\tilde{V}_C)\phi=\epsilon\phi,\quad \tilde{V}_C=\tfrac{1}{2}\omega^2\boldsymbol{x}^2\,, \qquad (3)$$

with energies ϵ and E_C related by [#1]

$$\epsilon=\epsilon(\omega)=\epsilon(\sqrt{-E_C/2})=Z\,. \qquad (4)$$

The main effect of this transformation can be summarized by the simple rule that the non-derivative terms are multiplied by a factor r: $V_C-E_C \to r(V_C-E_C)=-Z-E_C\boldsymbol{x}^2$. Using this rule, it is easy to see that the interaction term $V_I=g(x^2+y^2)$ of the Zeeman potential (assuming $L_z=0$, for simplicity) maps onto

$$\tilde{V}_I\propto\boldsymbol{x}^2(x_1^2+x_2^2)(x_3^2+x_4^2)\,, \qquad (5)$$

suggesting that the Zeeman Hamiltonian is related with an anharmonic anisotropic oscillator in four dimensions.

Of course, in order to establish this relation rigorously, a more careful derivation is necessary, which proceeds as follows. Separating the Schrödinger equation $H\psi=E\psi$ for the Zeeman Hamiltonian (1) in cylindrical coordinates ($x=\rho\cos\alpha$, $y=\rho\sin\alpha$, $z=z$) with $\psi=\rho^{-1/2}\varphi\exp(\mathrm{i}l_C\alpha)$, $l_C=0,\pm1,\ldots$, and changing then to parabolic coordinates ρ_1,ρ_2 defined by $z=\rho_1^2-\rho_2^2$, $\rho=2\rho_1\rho_2$, we arrive at ($Z\equiv1$)

$$\Big(-\tfrac{1}{2}(\partial_{\rho_1}^2+\partial_{\rho_2}^2)+\frac{l_C^2-\frac{1}{4}}{2\rho_1^2}+\frac{l_C^2-\frac{1}{4}}{2\rho_2^2}$$
$$+(-4\tilde{E})(\rho_1^2+\rho_2^2)+4g\times4(\rho_1^2+\rho_2^2)\rho_1^2\rho_2^2\Big)\varphi$$
$$=4Z\varphi\,, \qquad (6)$$

where $\tilde{E}\equiv E-Bl_C/2$. We now observe that this can be interpreted as the Schrödinger equation of two coupled two-dimensional anharmonic oscillators in cylindrical polar coordinates subject to the constraint

$$l_O^{(1)}=l_O^{(2)}=l_C\,. \qquad (7)$$

Hence, going back to Cartesian coordinates ($x_1=\rho_1\cos\alpha_1$, $x_2=\rho_1\sin\alpha_1$, $x_3=\rho_2\cos\alpha_2$, $x_4=\rho_2\sin\alpha_2$), we recover after a further trivial rescaling the heuristic result (5), but now supplemented with the "selection rule" (7) and with the BL_z-term taken into account:

$$h\phi\equiv[-\tfrac{1}{2}\nabla^2+\tfrac{1}{2}\omega^2\boldsymbol{x}^2$$
$$+4\lambda\boldsymbol{x}^2(x_1^2+x_2^2)(x_3^2+x_4^2)]\phi=\epsilon\phi\,, \qquad (8)$$

where

$$\omega^2\equiv-\tilde{E}/2,\quad \lambda\equiv g/4^3\,, \qquad (9)$$

and the energies ϵ and $\tilde{E}\equiv E-Bl_C/2$ are related by

$$\epsilon(\omega,\lambda)=\epsilon(\sqrt{-\tilde{E}/2},g/4^3)=Z\equiv1\,. \qquad (10)$$

As far as the energies are concerned this is the complete answer. To relate also the wave-functions in a

[#1] To be precise, this has to be supplemented by some constraints on the four-dimensional system [2], which we shall neglect for the moment

unique way, further constraints on the four-dimensional system are necessary which shall be discussed elsewhere [17].

As an application we shall now relate the perturbation expansions of the Zeeman energies,

$$\tilde{E}=\sum_{k=0} E_k g^k , \tag{11}$$

with those of the oscillator energies in (8) which, by a simple scaling argument, must be of the form

$$\epsilon=\omega \sum_{k=0} \epsilon_k(\lambda/\omega^4)^k . \tag{12}$$

If this is inserted in eq. (10), a perturbative solution yields the coefficients E_k expressed in terms of ϵ_l with $l\leqslant k$,

$$E_0=-2/\epsilon_0^2, \quad E_1=4\epsilon_0\epsilon_1/4^3 ,$$
$$E_2=\epsilon_0^4(10\epsilon_1^2+4\epsilon_2\epsilon_0)/4^6, \quad \ldots . \tag{13}$$

Identical relations emerge in the simplified model study [12] for isotropic perturbations $\propto r^p$ with $p=2$ in (1), corresponding to anharmonic terms $\propto |x|^{2(p+1)}$ in (8).

For the ground-state energy of (8), the low-order coefficients ϵ_k are easily calculated by means of standard perturbation theory,

$$\epsilon_0=2, \quad \epsilon_1=16, \quad \epsilon_2=-4^2\times\frac{166}{3}, \quad \ldots . \tag{14}$$

Inserting these numbers in (13), we recover the known coefficients for the ground-state expansion ($l_C=0$) of the Zeeman system [10,11],

$$E_0=-1/2, \quad E_1=2, \quad E_2=-53/3, \quad \ldots . \tag{15}$$

In principle, the perturbative solution of (10) can be extended to any order [12,17]. In large orders, however, this becomes quite cumbersome and it is advisable to use a method [#2] which is more specific to the large-order behaviour of these series. For anharmonic oscillator systems it is generically of the form [6,7] [#3]

$$\epsilon_k \xrightarrow{k\to\infty} -\frac{p}{\pi}\gamma(-a)^k\Gamma(pk+b) , \tag{16}$$

where γ and a equal some constants, the parameter p is related with the degree of anharmonicity ($\propto x^{2(p+1)}$, yielding $p=2$ in (8)), and b is associated with the symmetries of the system. An equivalent statement is that, for *negative* coupling, the energies develop an imaginary part of the form

$$\operatorname{Im}\epsilon(\omega,\lambda) \xrightarrow{\lambda\to 0-} \omega\gamma[(a|\lambda|/\omega^{2+p})^{-1/p}]^b \times\exp[-(a|\lambda|/\omega^{2+p})^{-1/p}] . \tag{17}$$

The equivalence of (16) and (17) can be shown by means of a dispersion relation [6,21],

$$\frac{\omega\epsilon_k}{(\omega^{2+p})^k}=\frac{1}{\pi}\int_{-\infty}^{0} d\lambda\,\frac{\operatorname{Im}\epsilon(\omega,\lambda+i0)}{\lambda^{k+1}} , \tag{18}$$

which has been proved rigorously for simple oscillator systems. The idea is now to relate first the imaginary parts of ϵ and $\tilde{E}$ by solving eq. (10) for small negative coupling. Using (18), this leads then immediately to the desired relation between the large-order coefficients. For general p and negative coupling, eq. (10) can be written as

$$1=\epsilon(\omega,\lambda)=\omega\sum_{k=0}\epsilon_k\left(\frac{\lambda}{\omega^{2+p}}\right)^k+i\operatorname{Im}\epsilon(\omega,\lambda) , \tag{19}$$

with $\omega^2=-\tilde{E}/2$ and $\lambda=g/4^{p+1}$. Here we have added the real part of ϵ which, perturbatively, has the same expansion as for $\lambda>0$ (see (12), extended to general p). To solve this for $\tilde{E}=-2\omega^2$, we put $\omega=x+iy$ and note that $y\propto\operatorname{Im}\epsilon$ is exponentially small. A straightforward perturbative solution [12] yields then to lowest order

$$\operatorname{Im}\tilde{E}(g) \xrightarrow{g\to 0-} \gamma^*[(a^*|g|)^{-1/p}]^b \times\exp[(a^*|g|)^{-1/p}] , \tag{20}$$

with [#4]

$$a^*=(\epsilon_0^{2+p}/4^{1+p})a, \quad \gamma^*=(4/\epsilon_0^3)\gamma . \tag{21}$$

We have thus succeeded in expressing the large-order parameters for quite general Coulomb systems in terms of those for the equivalent oscillator systems.

To complete our discussion for the Zeeman sys-

[#2] A similar approach was used in the field-theoretical context of ϵ-expansions for critical exponents in ref. [18].

[#3] For reviews, see ref. [19]. See also the forthcoming introductory text [20].

[#4] For $p=1$, γ^* has to be multiplied by a factor $\exp(-\epsilon_1/\epsilon_0)$, resulting from higher order corrections. For more details, see ref. [12].

tem, we now present a derivation of these parameters for the ground-state energy of the anisotropic anharmonic oscillator (8). By making use of known results for isotropic anharmonic oscillators, this turns out to be quite straightforward. We shall use the path-integral approach in Langer's formulation [22] (Langer's approach is somewhat different from the path-integral formulation in ref. [23]) which amounts to a saddle-point approximation, for small $\lambda<0$, of the imaginary part of the partition function

$$Z=\int \mathscr{D}^2u\,\mathscr{D}^2v\exp(-\mathscr{A}[\boldsymbol{u},\boldsymbol{v}])\overset{\beta\to\infty}{\longrightarrow} \mathrm{e}^{-\beta\epsilon}, \tag{22}$$

where

$$\mathscr{A}[\boldsymbol{u},\boldsymbol{v}]=\int_{-\beta/2}^{\beta/2}\mathrm{d}\tau\,[\tfrac{1}{2}\dot{\boldsymbol{u}}^2+\tfrac{1}{2}\dot{\boldsymbol{v}}^2+\tfrac{1}{2}\boldsymbol{u}^2+\tfrac{1}{2}\boldsymbol{v}^2 +4\lambda(\boldsymbol{u}^2+\boldsymbol{v}^2)\boldsymbol{u}^2\boldsymbol{v}^2] \tag{23}$$

is the Euclidean action associated with the Hamiltonian (8). Here we have introduced the convenient notation $\boldsymbol{x}=(\boldsymbol{u},\boldsymbol{v})=(u_1,u_2,v_1,v_2)$ and put $\omega=1$, for simplicity. The imaginary part of the ground-state energy ϵ (for $\lambda<0$) follows then from

$$\mathrm{Im}\,\epsilon=-\beta^{-1}\,\mathrm{Im}\,Z/\mathrm{Re}\,Z\,. \tag{24}$$

The real part of Z, $\mathrm{Re}\,Z=\exp\{-2\beta[1+\mathrm{O}(\lambda)]\}$, is caused by fluctuations around the locally stable minimum at $(\boldsymbol{u},\boldsymbol{v})=0$. Here and in the sequel, the large β limit is always implied.

The saddle points of (23) correspond to "tunneling paths" with least action. In our case they are obviously along the diagonal ray $|\boldsymbol{u}|=|\boldsymbol{v}|>0$. The directions of the two-dimensional vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are independent and can be choosen both arbitrarily, reflecting the $\mathrm{O}(2)\times\mathrm{O}(2)$ symmetry of the system. A convenient parametrization is

$$\boldsymbol{u}=\frac{1}{\sqrt{2}}(w,0)=\boldsymbol{v}, \tag{25}$$

since the action then reduces to that of a one-dimensional sextic anharmonic oscillator with the usual normalizations,

$$\mathscr{A}=\int\mathrm{d}\tau\,(\tfrac{1}{2}\dot{w}^2+\tfrac{1}{2}w^2+\lambda w^6)\,. \tag{26}$$

It is known [7,19] that this is further extremized by the "critical bubble" solution

$$w_{\mathrm{c}}(\tau)=\left(\frac{1}{2|\lambda|}\right)^{1/4}\frac{1}{\{\cosh[2(\tau-\tau_0)]\}^{1/2}}, \tag{27}$$

with associated action

$$\mathscr{A}_{\mathrm{c}}\equiv A/|\lambda|^{1/2}\equiv 1/(a|\lambda|)^{1/2}, \quad a=(1/A)^2=32/\pi^2\,. \tag{28}$$

Besides the directions of $\boldsymbol{u}$ and $\boldsymbol{v}$ also the origin τ_0 in (27) can be chosen arbitrarily. (In the sequel we shall put $\tau_0=0$.) Any choice breaks the symmetry spontaneously and leads to Nambu–Goldstone zero-modes which require a special treatment when considering the fluctuations around the "critical bubble" solution.

The leading contribution of such fluctuations is found by expanding the full action (23) in the deviations $\delta\boldsymbol{u}\equiv\boldsymbol{u}-\boldsymbol{u}_{\mathrm{c}}$, $\delta\boldsymbol{v}\equiv\boldsymbol{v}-\boldsymbol{v}_{\mathrm{c}}$ up to quadratic order,

$$\delta\mathscr{A}\equiv\mathscr{A}-\mathscr{A}_{\mathrm{c}}=\tfrac{1}{2}\int\mathrm{d}\tau\,[\delta\dot{\boldsymbol{u}}^2+\delta\dot{\boldsymbol{v}}^2+\delta\boldsymbol{u}^2+\delta\boldsymbol{v}^2 +8\lambda(\delta u_i\,\delta v_i)M_{ij}(\delta u_j\,\delta v_j)^{\mathrm{t}}+\ldots]\,, \tag{29}$$

where the superscript t denotes transposition. If the "critical bubble" solution is inserted, the 2×2 matrix M_{ij} can be decomposed into a longitudinal and transversal part,

$$(\delta u_i\,\delta v_i)M_{ij}(\delta u_j\,\delta v_j)^{\mathrm{t}}=(\delta u_1\,\delta v_1)M_{\mathrm{L}}(\delta u_1\,\delta v_1)^{\mathrm{t}} +(\delta u_2\,\delta v_2)M_{\mathrm{T}}(\delta u_2\,\delta v_2)^{\mathrm{t}}, \tag{30}$$

with

$$M_{\mathrm{L}}=(w_{\mathrm{c}}/\sqrt{2})^4\begin{pmatrix}7&8\\8&7\end{pmatrix}, \quad M_{\mathrm{T}}=(w_{\mathrm{c}}/\sqrt{2})^4\begin{pmatrix}3&0\\0&3\end{pmatrix}. \tag{31}$$

Diagonalizing the matrix M_{L} (with eigenvalues $M_{\mathrm{L}}^{(1,2)}=(w_{\mathrm{c}}/\sqrt{2})^4(-1,15)$) by a 45° rotation to new coordinates

$$\xi=\frac{1}{\sqrt{2}}(\delta u_1+\delta v_1), \quad \eta=\frac{1}{\sqrt{2}}(\delta u_1-\delta v_1)\,,$$

the longitudinal part of $\delta\mathscr{A}=\delta\mathscr{A}_{\mathrm{L}}+\delta\mathscr{A}_{\mathrm{T}}$ decouples into

$$\delta\mathcal{A}_L=\frac{1}{2}\int d\tau\left[\xi\left(-\frac{d^2}{d\tau^2}+1-\frac{15}{\cosh^2 2\tau}\right)\xi\right.$$
$$\left.+\eta\left(-\frac{d^2}{d\tau^2}+1+\frac{1}{\cosh^2 2\tau}\right)\eta+\ldots\right]. \quad (32)$$

The operator governing the ξ fluctuations coincides precisely with that of the one-dimensional x^6 oscillator. Its associated fluctuation determinant, resulting from the Gaussian integrations over ξ, can therefore be taken directly from the literature [7,19]:

$$f_\xi\equiv\det\left(-\frac{d^2}{d\tau^2}+1-\frac{15}{\cosh^2 2\tau}\right)^{-1/2}$$
$$=-\frac{i}{2}\frac{1}{\sqrt{2^{-1/2}\pi A}}\sqrt{\mathcal{A}_c}\,\beta e^{-\beta/2}. \quad (33)$$

The imaginary unit, indicating the expected metastability, is caused by an eigenmode ξ_- with *negative* eigenvalue, and the factor β is associated with a zero-mode $\xi_0\propto\dot{w}_c$, reflecting the translational invariance in τ, i.e., the freedom of choosing τ_0 in (27).

The operator associated with the η fluctuations obviously has only positive eigenvalues whose product can be calculated as follows. Separating out the harmonic fluctuations,

$$f_\eta=\det\left(-\frac{d^2}{d\tau^2}+1+\frac{1}{\cosh^2 2\tau}\right)^{-1/2}\equiv f Z_{osc}, \quad (34)$$

where

$$Z_{osc}\equiv\det\left(-\frac{d^2}{d\tau^2}+1\right)^{-1/2}$$
$$=\frac{1}{2\sinh(\beta/2)}\xrightarrow{\beta\to\infty}e^{-\beta/2}$$

is the partition function of the harmonic oscillator, and changing variables to $t=2\tau$, we obtain

$$f\equiv\left(\frac{\det[-d^2/dt^2+z-s(s+1)/\cosh^2 t]}{\det(-d^2/dt^2+z)}\right)^{-1/2}$$
$$=\left(\frac{\Gamma(\sqrt{z}-s)\Gamma(\sqrt{z}+1+s)}{\Gamma(\sqrt{z})\Gamma(\sqrt{z}+1)}\right)^{1/2}, \quad (35)$$

with $z=1/4$ and $s=-1/2$. The second line follows from a general formula [7,19] in the theory of Fredholm determinants which, in more physical terms, can be derived [20] by relating f to the quantum-mechanical transmission amplitude of one-dimensional scattering at the potential $z-s(s+1)/\cosh^2 t$. Inserting the numbers for z and s, we finally have

$$f_\eta=(2/\pi)^{1/2}e^{-\beta/2}. \quad (36)$$

The transversal fluctuation factors can be taken again from the literature on anharmonic oscillators [7,19]. Since the matrix M_T is proportional to unity, the transversal degrees of freedom of $\delta\boldsymbol{u}$ and $\delta\boldsymbol{v}$ decouple automatically, and we find

$$\delta\mathcal{A}_T=\frac{1}{2}\int d\tau\left[\delta u_2\left(-\frac{d^2}{d\tau^2}+1-\frac{3}{\cosh^2 2\tau}\right)\delta u_2\right.$$
$$\left.\times(\delta u_2\leftrightarrow\delta v_2)+\ldots\right], \quad (37)$$

with an operator identical to that governing the transversal fluctuations of a general $O(n)$ symmetric $|\boldsymbol{x}|^6$ oscillator. It contains one zero-mode $\propto w_c$ with eigenvalue $\kappa_0=0$, associated with the freedom of choosing the direction of $\boldsymbol{u}_c$ and $\boldsymbol{v}_c$, respectively. Adapting the general prescription [7,19] for dealing with such rotational zero-modes to our case $n=2$, we obtain

$$(1/\kappa_0)^{1/2}\to S_2\left(\int d\tau\frac{\boldsymbol{u}_c^2}{2\pi}\right)^{1/2}, \quad (38)$$

where $S_2=2\pi$ is the "surface" of the unit-circle. Inserting $\boldsymbol{u}_c^2=\frac{1}{2}w_c^2$ and using the scaling properties of the action, the square root factor can be simplified to

$$\left(\int d\tau\frac{\boldsymbol{u}_c^2}{2\pi}\right)^{1/2}=\frac{1}{\sqrt{2}}\left(\int d\tau\frac{w_c^2}{2\pi}\right)^{1/2}$$
$$=\frac{1}{\sqrt{2}}\sqrt{\mathcal{A}_c/\pi}. \quad (39)$$

Notice the additional factor $1/\sqrt{2}$ compared with the ordinary O(2)-symmetric $|\boldsymbol{x}|^6$ oscillator. Taking into account the known contribution [7,19] of all other modes with positive eigenvalues, we get

$$f_{\delta u_2}=f_{\delta v_2}=\frac{1}{\sqrt{2}}\left(\frac{1}{2^{-1/2}A}\right)^{1/2}$$
$$\times 2\pi\sqrt{\mathcal{A}_c/\pi}\,e^{-\beta/2}. \quad (40)$$

Finally, combining (33), (36) and (40) we find

$$\mathrm{Im}\,Z=-|f_\xi|f_\eta f_{\delta u_2}f_{\delta v_2}\exp(-\mathcal{A}_c),$$

and using (24) we end up with

$$\operatorname{Im}\epsilon=\sqrt{2}\left(\frac{1}{2^{-1/2}A}\right)^{3/2}\mathscr{A}_c^{3/2}\exp(-\mathscr{A}_c)$$

$$=\frac{32}{\pi^{3/2}}\mathscr{A}_c^{3/2}\exp(-\mathscr{A}_c)\,, \tag{41}$$

where $\mathscr{A}_c=(32|\lambda|/\pi^2)^{-1/2}$ (see eq. (28)). Recalling the dispersion relation (18), this is equivalent to

$$\epsilon_k \overset{k\to\infty}{\longrightarrow} -\frac{64}{\pi^{5/2}}(-32/\pi^2)^k\Gamma(2k+3/2)\equiv\epsilon_k^{\mathrm{asy}}\,, \tag{42}$$

displaying directly the large-order parameters p, a, b, γ defined in (16):

$$p=2,\quad a=32/\pi^2,\quad b=3/2,\quad \gamma=32/\pi^{3/2}\,. \tag{43}$$

Using (21) this implies for the Zeeman ground-state energy the large-order formula

$$E_k \overset{k\to\infty}{\longrightarrow} -\frac{32}{\pi^{5/2}}(-8/\pi^2)^k\Gamma(2k+3/2)\equiv E_k^{\mathrm{asy}}\,, \tag{44}$$

in agreement with the calculations in ref. [11] based on multidimensional WKB techniques. Notice that in the present path-integral approach the leading behaviour, $E_k\propto\Gamma(2k+3/2$, can be understood by simple scaling ($p=2$) and symmetry ($b=$ half the number of zero-modes $=3/2$) arguments alone.

The asymptotic formula (42) has been checked against exact coefficients in high order (up to $k=100$) calculated by means of special recursion relations [17]. Applying numerical (Neville-like) extrapolation schemes [24] to this series, the correction parameters γ_1, γ_2 in the ansatz

$$\epsilon_k=\epsilon_k^{\mathrm{asy}}\left(1+\frac{\gamma_1}{k}+\frac{\gamma_2}{k^2}+\ldots\right) \tag{45}$$

can be calculated quite accurately and then transformed back to the Zeeman system. It turns out [17] that the oscillator series approaches its asymptotic behaviour much more rapidly ($\gamma_1=-0.15090$, $\gamma_2=-3.9821$) than the Zeeman series ($\gamma_1^*=-2.6183$, $\gamma_2^*=1.2847$). This observation might be important for efficient use of resummation algorithms (like, e.g., Borel's [19]) which make use of the large-order information. Moreover, the strong coupling behaviour of the oscillator energies ($\epsilon\propto\lambda^{1/4}$ as $\lambda\to\infty$) is more regular than that of the Zeeman energies (containing terms $\propto\log^2 g$, $\log(\log g)$ as $g\to\infty$ [25]), which should further improve the performance of such algorithms. This suggests that the new equivalence may lead to improved numerical schemes for calculating precise Zeeman energies from perturbation theory. Furthermore, it is clear that many techniques available for the oscillator system (such as rigorous bounds etc.) are transferable to the Zeeman system. It would be interesting to investigate these points in more detail.

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