High-temperature series analysis of the free energy and susceptibility of the 2D random-bond Ising model

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Abstract

We derive high-temperature series expansions for the free energy and susceptibility of the two-dimensional random-bond Ising model with a symmetric bimodal distribution of two positive coupling strengths J_1 and J_2 and study the influence of the quenched, random bond-disorder on the critical behavior of the model. By analysing the series expansions over a wide range of coupling ratios J_2/J_1, covering the crossover from weak to strong disorder, we obtain for the susceptibility with two different methods compelling evidence for a singularity of the form $\chi \sim t^{-7/4}|\ln t|^{7/8}$, as predicted theoretically by Shalaev, Shankar, and Ludwig. For the specific heat our results are less convincing, but still compatible with the theoretically predicted log–log singularity. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

One of the most-studied variants of the two-dimensional Ising model is the case of random bonds. While realizations of Ising models that include randomness come much closer to approximating reality, they are very much harder to study at any level. Even in two dimensions exact results for random cases (especially for quenched randomness, which is the realistic situation in many experiments) are few and far-between. In fact, the two-dimensional Ising case is especially difficult because of the marginality of the Harris criterion [1] for this model. This criterion states that quenched randomness is a relevant (irrelevant) perturbation when the critical exponent α of the specific heat of the

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pure system is positive (negative) and therefore when $\alpha = 0$ as in the two-dimensional
Ising model the situation is marginal.

Numerous theoretical investigations [2–10] as well as numerical Monte Carlo sim-
ulations [11–19] and transfer-matrix studies [20,21] have addressed the question of
whether the critical exponents for the two-dimensional Ising model with quenched, ran-
dom bond disorder differ from those of the pure model. While a "majority" consensus
had probably been achieved in favor of no change, apart from logarithmic corrections
[4–9], no unambiguous numerical study that confirmed the quantitative predictions of
either of the theoretical approaches had been made prior to our recent study of the sus-
ceptibility with high-temperature series expansions. In a brief note [22] we announced
the confirmation of the theoretical majority consensus value of the exponent of the
logarithmic correction. This quantitative determination of the value of the correction
exponent in excellent agreement with the predicted value, using a completely different
numerical approach that in no way depends on random numbers, provided additional
and incontrovertible support to the previous consensus. In the present paper, we present
the coefficients of the susceptibility series that we analyzed [22] together with some
remarks on their derivation, the details of our analysis, and some additional results for
the specific-heat series. We note that a recent simulation of the site-diluted model [23]
also confirms the log–log prediction of [4–9].

In the next section, we define the model and the quantities that are studied, and
in Section 3 the theoretical predictions are briefly recalled. The series generation is
described in Section 4, and in Section 5 we describe the analysis techniques used.
Section 6 then presents the results, where details of our analyses for the susceptibility
give compelling evidence for a singularity of the form predicted by Shalaev, Shankar,
and Ludwig [4–9]. In Section 7 we close with a summary of our conclusions and a
few final comments.

2. Model

The Hamiltonian of the random-bond Ising model is given by

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j \,, \tag{1}$$

where the spins $\sigma_i = \pm 1$ are located at the sites of a square lattice, the symbol $\langle ij \rangle$
denotes nearest-neighbor interactions, and the coupling constants J_{ij} are quenched, ran-
dom variables. As in most previous studies we consider a bimodal distribution,

$$P(J_{ij}) = x\delta(J_{ij} - J_1) + (1-x)\delta(J_{ij} - J_2) \,, \tag{2}$$

of two ferromagnetic couplings $J_1, J_2 > 0$. We furthermore specialize to a symmetric
distribution with $x = \frac{1}{2}$, since in this case the exact critical temperature T_c can be
computed for any positive value of J_1 and J_2 from the (transcendental) self-duality
relation (k_B = Boltzmann constant) [24,25].

$$(\exp(2J_1/k_B T_c) - 1)(\exp(2J_2/k_B T_c) - 1) = 2 \,. \tag{3}$$

In both, computer simulations and high-temperature series expansion studies, this exact information simplifies the analysis of the critical behavior considerably.

The free energy per site is given by

$$\beta f = - \lim_{V \to \infty} \frac{1}{V} \left[\ln \left(\prod_{i=1}^{V} \sum_{\sigma_i = \pm 1} \right) \exp(-\beta \mathscr{H}) \right]_{av} , \tag{4}$$

where $\beta = 1/k_B T$ is the inverse temperature in natural units and the bracket $[\ldots]_{av}$ denotes the average over the quenched, random disorder, $[\ldots]_{av} = (\prod_{\langle ij \rangle} \int dJ_{ij})(\ldots)P(J_{ij})$. The internal energy and specific heat per site follow by differentiation with respect to β,

$$e = \partial \beta f / \partial \beta, \qquad C/k_B = -\beta^2 \partial^2 \beta f / \partial \beta^2 . \tag{5}$$

In this paper, we shall mainly focus on the magnetic susceptibility per site χ which in the high-temperature phase and zero external field is defined as the $V \to \infty$ limit of

$$\chi_V = \left[\left\langle \left(\sum_{i=1}^{V} \sigma_i \right)^2 \right\rangle_T \Big/ V \right]_{av} , \tag{6}$$

where $\langle \ldots \rangle_T$ denotes the usual thermal average with respect to $\exp(-\beta \mathscr{H})$.

3. Theoretical predictions

Let us briefly recall two contradicting theoretical predictions for the critical behavior of the model (1) and (2). The first is based on renormalization-group techniques developed by Dotsenko and Dotsenko (DD) [2,3]. For the specific heat DD find close to the transition point a double logarithmic behavior,

$$C(t) \propto \ln(\ln(1/|t|)) , \tag{7}$$

where $t = (T - T_c)/T_c$ denotes the reduced temperature. For the susceptibility they obtain in the high-temperature phase $(t \geqslant 0)$

$$\chi \propto t^{-2} \exp \left[-a \left(\ln \ln \left(\frac{1}{t} \right) \right)^2 \right] . \tag{8}$$

The second approach by Shalaev, Shankar, and Ludwig (SSL) [4–9] makes use of bosonization techniques and the method of conformal invariance. While the prediction (7) for the specific heat can be reproduced (which, however, is not undisputed [10]), SSL derive quite a different behavior for the susceptibility,

$$\chi \propto t^{-7/4} |\ln t|^{7/8} . \tag{9}$$

This is the same leading singularity as in the pure case $(J_1 = J_2)$, but modified by a multiplicative logarithmic correction.

High-precision Monte Carlo simulations and transfer-matrix studies [11–20] favor the latter form, but due to well-known inherent limitations of this method it has been

Table 1
Coefficients a_n of the high-temperature series expansion of the free energy per site, $-\beta f = \ln[2\cosh(\beta J_1)\cosh(\beta J_2)] + \sum_n a_n k^n$, with $k = 2\beta J_1$

n	$J_2/J_1 = 1$	$J_2/J_1 = 2$	$J_2/J_1 = 4$	$J_2/J_1 = 10$
4	$\frac{1}{16}$	$\frac{81}{256}$	$\frac{625}{256}$	$\frac{14641}{256}$
6	$\frac{1}{96}$	$\frac{81}{2048}$	$-\frac{18125}{6144}$	$-\frac{5343965}{6144}$
8	$\frac{17}{2560}$	$\frac{33671}{327680}$	$-\frac{26293}{65536}$	$-\frac{990135929}{327680}$
10	$\frac{1907}{483840}$	$\frac{3437297}{27525120}$	$\frac{1057390637}{49545216}$	$\frac{16514750542133}{49545216}$

impossible to confirm the value of the exponent of the multiplicative logarithmic correction in (9) quantitatively. Similar problems have been reported in simulation studies of other models exhibiting multiplicative logarithmic corrections such as, e.g., the two-dimensional four-state Potts [26] and XY [27–30] model. We found it therefore worthwhile to investigate this problem yet again with an independent method. In the following we report high-temperature series expansions for the free energy and susceptibility and enquire whether series analyses can yield a more stringent test of the theoretical predictions.

4. Series generation

For the generation of the high-temperature series expansions of the free energy (4), and hence the internal energy and specific heat, as well as the infinite-volume limit of the susceptibility (6) we made use of a program package developed at Mainz originally for the q-state Potts spin–glass problem [31–36]. In this application the spin–spin interaction is generalized from $\sigma_i \sigma_j$ to $\delta_{\sigma_i, \sigma_j}$ with σ_i being an integer between 1 and q, and the coupling constants J_{ij} can take the values $\pm J$ at random. Since here the coupling constants also can take negative values, frustration effect play an important role and the physical properties of spin glasses [37] are completely different than those of the random-bond system. Technically, however, precisely the same enumeration scheme for the high-temperature graphs can be employed in both cases. The only difference is in the last step where the quenched averages over the J_{ij} are performed . The details of the employed star-graph expansion technique and our specific implementation are described elsewhere [31–34,36,38]. Here we only note that slight modifications of this program package enabled us to generate the high-temperature series expansion for βf and χ up to the 11th order in $k = 2\beta J_1$ for
- hypercubic lattices of arbitrary dimension d,
- arbitrary number of Potts states q,
- arbitrary probability x in the bimodal distribution, and
- arbitrary ratios $R = J_2/J_1$, characterizing the strength of the disorder.

In this paper, we shall concentrate on the two-dimensional ($d = 2$) random-bond Ising model ($q = 2$) for a symmetric ($x = \frac{1}{2}$) bimodal distribution of two positive

Table 2
Coefficients b_n of the high-temperature series expansion of the susceptibility per
site, $\chi = 1 + \sum_n b_n k^n$, with $k = 2\beta J_1$

n	$J_2/J_1 = 1$	$J_2/J_1 = 2$	$J_2/J_1 = 4$	$J_2/J_1 = 10$
1	2	3	5	11
2	3	$\frac{27}{4}$	$\frac{75}{4}$	$\frac{363}{4}$
3	$\frac{13}{3}$	$\frac{231}{16}$	$\frac{3115}{48}$	$\frac{31933}{48}$
4	$\frac{23}{4}$	$\frac{1809}{64}$	$\frac{13025}{64}$	$\frac{277937}{64}$
5	$\frac{451}{60}$	$\frac{69337}{1280}$	$\frac{471185}{768}$	$\frac{101248147}{3840}$
6	$\frac{191}{20}$	$\frac{515871}{5120}$	$\frac{1823875}{1024}$	$\frac{768499919}{5120}$
7	$\frac{30283}{2520}$	$\frac{79576207}{430080}$	$\frac{1302083479}{258048}$	$\frac{1034056024661}{1290240}$
8	$\frac{100003}{6720}$	$\frac{191638233}{573440}$	$\frac{4823704415}{344064}$	$\frac{7079050432267}{1720320}$
9	$\frac{3318601}{181440}$	$\frac{587805509}{983040}$	$\frac{203928262469}{5308416}$	$\frac{3850544162365417}{185794560}$
10	$\frac{3369629}{151200}$	$\frac{48645511629}{45875200}$	$\frac{5160699783175}{49545216}$	$\frac{126985060534491247}{1238630400}$
11	$\frac{269543489}{9979200}$	$\frac{101837138460677}{54499737600}$	$\frac{9157142004160957}{32699842560}$	$\frac{1069481408075459203}{2123366400}$

coupling strengths J_1 and J_2. The series coefficients of the free energy and susceptibility
expansions for various coupling-strength ratios $R = J_2/J_1$ are given in Tables 1 and 2.
Of course, in principle it would be also straightforward to adapt the present program
package to more general probability distributions $P(J_{ij})$.

5. Series analysis techniques

In the literature, many different series analysis techniques have been discussed which,
depending on the type of critical singularity at hand, all have their own merits and
drawbacks [39]. In the course of this work we have tested quite a few of them [39,38].
Here, however, we will confine ourselves to only those which turned out to be the most
useful for our specific problem at hand.

To simplify the notation we denote a thermodynamic function generically by $F(z)$
and assume that its Taylor expansion around the origin is known up to the Nth order,

$$F(z) = \sum_{n=0}^{N} a_n z^n + \cdots . \tag{10}$$

If the singularity of $F(z)$ at the critical point z_c is of the simple form ($z \leqslant z_c$)

$$F(z) \simeq A(1 - z/z_c)^{-\lambda} , \tag{11}$$

with A being a constant, then the ratios of consecutive coefficients approach for large
n the limiting behavior

$$r_n \equiv \frac{a_n}{a_{n-1}} \simeq \left[1 + \frac{\lambda - 1}{n} \right] \frac{1}{z_c} . \tag{12}$$

From the offset ($1/z_c$) and slope ($(\lambda - 1)/z_c$) of this sequence as a function of $1/n$ both
the critical point z_c and the critical exponent λ can be determined. This is the basis

of the so-called ratio method [40]. If the critical point z_c is known from other sources (in our case exactly from self-duality), then one may consider biased extrapolants for the critical exponent,

$$\lambda_n = n r_n z_c - n + 1 \, , \tag{13}$$

which simply follow by rearranging Eq. (12). In the following, this method will be denoted as "Biased Ratio I".

If the singularity of $F(z)$ contains a multiplicative logarithmic correction (as, e.g., in the SSL prediction for χ),

$$F(z) \simeq A(1 - z/z_c)^{-\lambda} |\ln(1 - z/z_c)|^p \, , \tag{14}$$

then one forms the ratios r_n as before, but considers in addition the auxiliary function [41]

$$z^{-p^*}(1 - z)^{-\lambda}(\ln[1/(1 - z)])^{p^*} = \sum_{n=0}^{N} b_n z^n + \cdots \, , \tag{15}$$

and computes the ratios $r_n^* = b_n/b_{n-1}$. Let us first assume that the critical exponent λ of the leading term is known. Then it can be shown that the sequence $R_n = r_n/r_n^*$ approaches $1/z_c$ with zero slope in the limit $n \to \infty$, if and only if $p^* = p$. This determines p, if also z_c is known. If λ is not known, then one may vary both exponents until the above relation is satisfied. In the following, we refer to this special ratio method as "Ln-Ratio".

Another method [42,43] suitable for a singularity of the form (14) is based on Padé approximants [44]. Here one generates the series expansion for the auxiliary function

$$G(z) = -(z_c - z) \ln(z_c - z) \left(\frac{F'(z)}{F(z)} - \frac{\lambda}{z_c - z} \right) , \tag{16}$$

which can easily be shown to satisfy

$$\lim_{z \to z_c} G(z) = p \, . \tag{17}$$

If z_c is known, the value of $G(z)$ at $z = z_c$ can be obtained by forming Padé approximants,

$$G(z) \approx [L/M] \equiv \frac{P_L(z)}{Q_M(z)} = \frac{p_0 + p_1 z + p_2 z^2 + \cdots + p_L z^L}{1 + q_1 z + q_2 z^2 + \cdots + q_M z^M} \, , \tag{18}$$

with $L + M \leqslant N - 1$. Note that one order of the initial series is lost due to the differentiation in (16).

With a small modification this method can also be applied to a purely logarithmic singularity of the form

$$F(z) \simeq A |\ln(1 - z/z_c)|^p \, . \tag{19}$$

In this case one defines the auxiliary function

$$G(z) = -(z_c - z) \ln(z_c - z) \frac{F'(z)}{F(z)} \, , \tag{20}$$
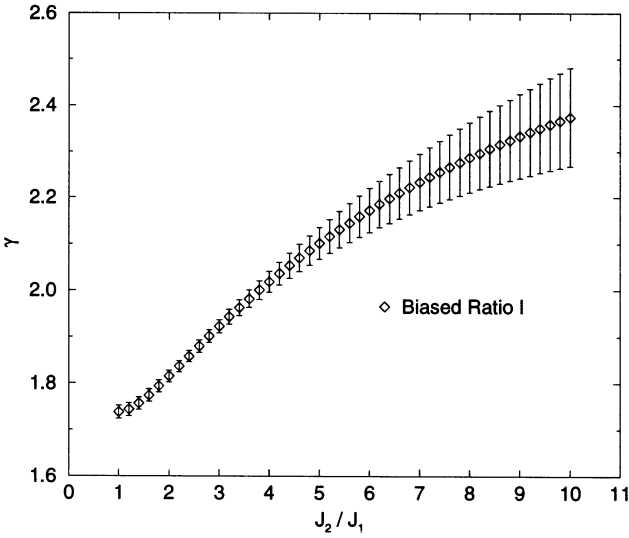
Fig. 1. Analysis of the susceptibility series assuming a singularity of the form $\chi \propto t^{-\gamma}$, using the method "Biased Ratio I".

which again satisfies

$$\lim_{z \to z_c} G(z) = p \,. \tag{21}$$

The two analysis methods based on Padé approximants will be called "Ln-Padé".

6. Results

6.1. Susceptibility

In a first step, we investigated whether our series expansions for the susceptibility are consistent with a pure power-law behavior according to the DD prediction (8) (ignoring the exponentially small multiplicative correction term). Assuming thus the behavior $\chi \propto t^{-\gamma}$ and using the method "Biased Ratio I" we obtained the critical exponents γ shown in Fig. 1 as a function of J_2/J_1. Here and in the following the error bars are estimated by varying the length of the series and/or the type of Padé approximants used. Starting with $\gamma = 1.738 \pm 0.014$ for the pure case ($J_2/J_1 = 1$), being consistent with the exact value of $\gamma = 7/4$, we observe a steady increase to $\gamma = 2.37 \pm 0.11$ for the strongest investigated disorder ($J_2/J_1 = 10$). We will argue below that the apparent crossover from weak to strong disorder is due to the finite length of our series expansion which naturally has a much more dramatic influence for weak disorder. At any rate, for strong disorder the DD prediction of $\gamma = 2$ is clearly outside the error margins of the series analysis estimates.
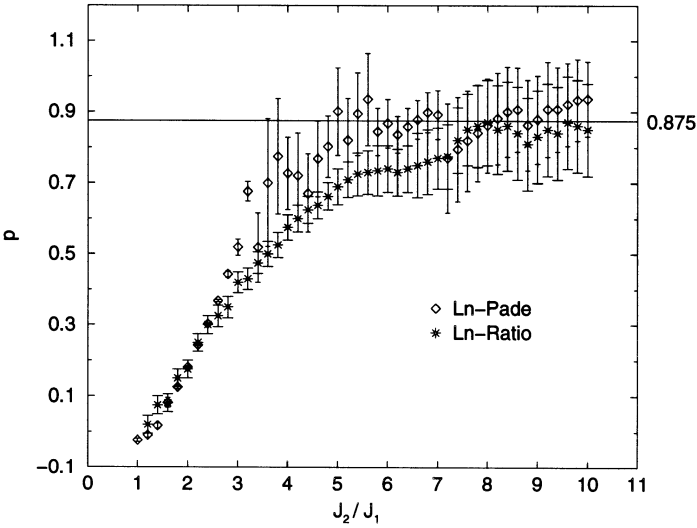
Fig. 2. Analysis of the susceptibility series assuming a singularity of the form $\chi \propto t^{-7/4}|\ln t|^p$, using Padé approximants and the ratio method (see text). The horizontal line at $p = 7/8 = 0.875$ is the theoretical prediction of SSL.

So far no multiplicative logarithmic corrections were taken into account. If the SSL prediction (9) was correct we would, therefore, expect to observe "effective" critical exponents which according to

$$\chi \propto t^{-7/4}|\ln t|^{7/8} = t^{-(7/4)[1+(1/2)\ln(|\ln t|)/\ln(1/t)]} \tag{22}$$

should indeed be larger than 7/4. The results in Fig. 1 could thus be well consistent with a critical exponent of $\gamma = 7/4$ in the presence of a multiplicative logarithmic correction.

This possibility suggested a more careful analysis based on the qualitative form of the SSL prediction (9). Our series are too short to employ a general ansatz with both exponents as free parameters. We rather fixed the exponent $\gamma = 7/4$ of the leading term to the (predicted) pure Ising model value and enquired if our series expansions are compatible with the ansatz

$$\chi \propto t^{-7/4}|\ln t|^p , \tag{23}$$

and $p = 7/8$. Employing the two special methods for this type of singularity described in Section 5 we obtained well converging results. The resulting estimates for the exponent p are shown in Fig. 2. We see that the two methods yield consistent results which start in the pure case ($J_2/J_1 = 1$) around $p = 0$, as they should do. With increasing disorder the estimates exhibit again an apparent crossover, until around $J_2/J_1 = 5 - 8$ they settle at a plateau value in very good agreement with the theoretical prediction of $p = 7/8$. This is the main result of our series analysis. We claim that compared with previous methods this is thus far the clearest quantitative confirmation of the SSL prediction (9).

As before we attribute the apparent crossover for intermediate strength of the disorder to the shortness of our series expansions, i.e., we interpret the crossover as an unavoidable artifact of high-temperature series expansion analyses and not as an indication that the exponent p really is a function of the disorder strength. We thus take the view that already a small amount of disorder drives the system into a new universality class different from the pure case which, however, only becomes visible in the very vicinity of the transition point T_c (or $t = 0$). This in turn translates into the need of extremely long series expansions in order to be detectable.

To justify this claim we have investigated a model function simulating the "true" susceptibility, where $g_0 \geqslant 0$ is a constant that depends on the strength of the disorder,

$$\chi_{\mathrm{model}} = i^{-7/4} \left[1 + \frac{4g_0}{\pi} \ln(1/i) \right]^{7/8}, \tag{24}$$

with $i = (T - T_c)/T$, which for any $g_0 \neq 0$ reproduces the SSL form (9) in the limit $T \to T_c$ ($i = t - t^2 + t^3 + \cdots \to 0$). Notice the discontinuity in the asymptotic behavior at $g_0 = 0$. For any $g_0 \neq 0$ the asymptotic region is reached when $\ln(1/t)$ is much larger than $\pi/4g_0$, i.e., for $t \ll \exp(-\pi/4g_0)$. Since $g_0 = 0$ corresponds to the pure case it is intuitively clear that the parameter g_0 is an increasing function of the degree of disorder. For weak disorder this implies that g_0 is very small and therefore, due to the exponential dependence, that the asymptotic region in t is extremely narrow.

Strictly speaking the model function (24) should resemble the "true" susceptibility only for weak disorder, but it is commonly believed that it is a reasonable qualitative approximation also for strong disorder. To relate the parameter g_0 at least heuristically to the ratio J_2/J_1 we used the weak disorder result $g_0 = c_2 a^2/(1 + ab)^2$, where $c_2 = 1 - x$ (with x as defined in Eq. (2)) is assumed to be small, $c_2 \ll 1$, i.e., the analytic calculation assumes that there are only few J_2-bonds in a background of J_1-bonds. The parameters a and b are given by $a = (v'_c - v_c^{(0)})/v_c^{(0)}$ and $b = v_c^{(0)}/2\sqrt{2}$, where $v_c^{(0)} = \tanh(\beta_c^{(0)} J_1) = \sqrt{2} - 1$ and $v'_c = \tanh(\beta_c^{(0)} J_2)$, with $\beta_c^{(0)}$ denoting the inverse critical temperature of the *pure* system with all $J_{ij} = J_1$. Of course, employing this formula to the present case with $c_2 = 1/2 = x$ is a bold step which even creates an ambiguity since the exact symmetry $J_1 \leftrightarrow J_2$ for $x = 1/2$ is violated. For weak disorder ($J_2/J_1 \approx 1$), however, the inconsistency turns out to be very mild. For $J_2/J_1 = 1.2$ we obtain $g_0 = 0.013700 \ldots$, and for $J_2/J_1 = 1/1.2$ we find a slightly smaller value of $g'_0 = 0.011958 \ldots$. This shows that for weak disorder ($J_2/J_1 = 1.2$, $g_0 \approx 0.013$) the asymptotic region is bounded by $t \ll \exp(-\pi/4g_0) \approx \exp(-1/0.017) \approx 10^{-26}$, and thus explains why it is so difficult to observe the asymptotic critical behavior in the weak-disorder limit. For $J_2/J_1 = 1.5$ and $1/1.5$ the corresponding numbers are $g_0 = 0.070700 \ldots$ and $g'_0 = 0.052889 \ldots$, leading to a bound of the order of $t \ll 10^{-5}$.

Using a symbolic computer program it is straightforward to generate the high-temperature series expansion of the model function (24) to any desired order. Applying the same analysis techniques as used for the "true" susceptibility series we obtained the results shown in Fig. 3. If we truncate the model series at low order we observe
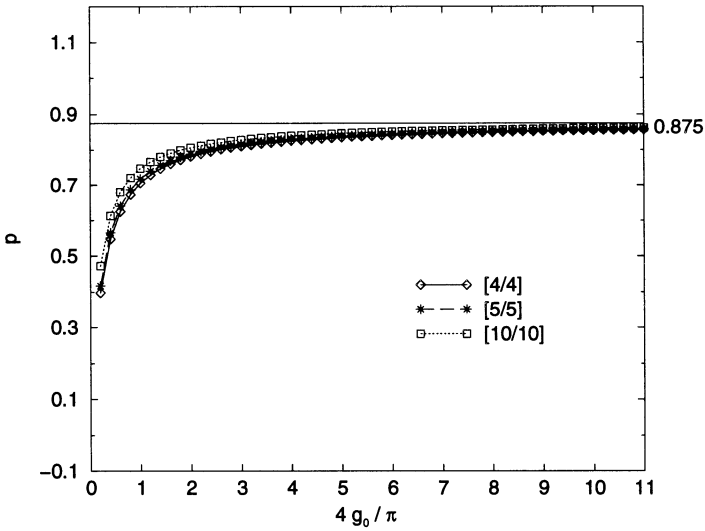
Fig. 3. Analysis of the series expansion of the model function (24), using the ansatz $\chi_{\mathrm{model}} \propto t^{-7/4}|\ln t|^p$. The legend indicates the three different Padé approximants shown. The horizontal line shows the exact value $p = 7/8 = 0.875$.

qualitatively the same crossover effect as for the "true" series. Here we are sure, however, that this must be a pure artifact of the truncation of the model series at a finite order. We also see that the approach of the asymptotic limit of $p = 7/8$ as a function of the degree of disorder is faster if we consider a longer series (21 terms). It is, however, somewhat discouraging (even though understandable in view of the exponential dependence of the critical regime on g_0) that at a fixed g_0 the convergence of the series with increasing order is quite slow. For example, at $4g_0/\pi = 1$ we obtained $p = 0.7056$ with the Padé approximant [4/4], 0.7178 ([5/5]), 0.7474 ([10/10]), 0.7682 ([20/20]), 0.7777 ([30/30]), 0.7834 ([40/40]), 0.7875 ([50/50]), and 0.7905 ([60/60]). The convergence behavior for this example and other small values of the parameter g_0 can be visually inspected in Fig. 4.

6.2. Specific heat

Series analyses for the specific heat are usually more difficult than for the susceptibility. This is especially pronounced for the Ising model on loose-packed lattices where all odd powers of β vanish because of symmetry. Consequently, our specific-heat series consists only of four nontrivial terms (see Table 1). We nevertheless tried an analysis with the ansatz

$$C \propto |\ln t|^q ,\tag{25}$$

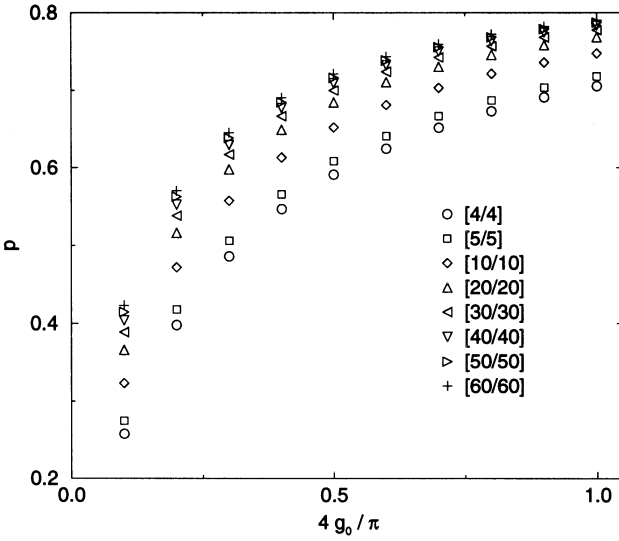using the method "Ln-Padé". The exponent q is an effective exponent whose value may, or may not be constant.

Fig. 4. Convergence behavior of the model series (24) for the susceptibility with increasing order at fixed parameter g_0. The symbols $[L/M]$ denote the various Padé approximants used.
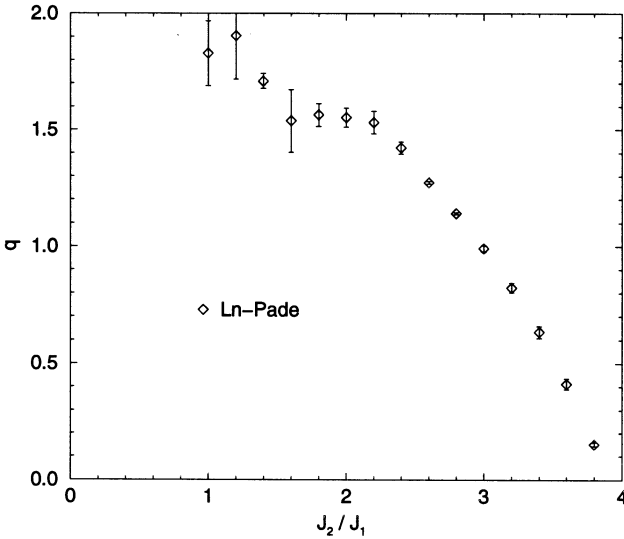

Fig. 5. Analysis of the specific-heat series using the Ln-Padé method.

The resulting dependence of the exponent q on the ratio J_2/J_1 is shown in Fig. 5. While the quantitative agreement with the exactly known pure case is certainly not convincing, we do see at least a qualitative trend to smaller values of q with increasing strength of the disorder (increasing ratios J_2/J_1), i.e., the singularity of the specific
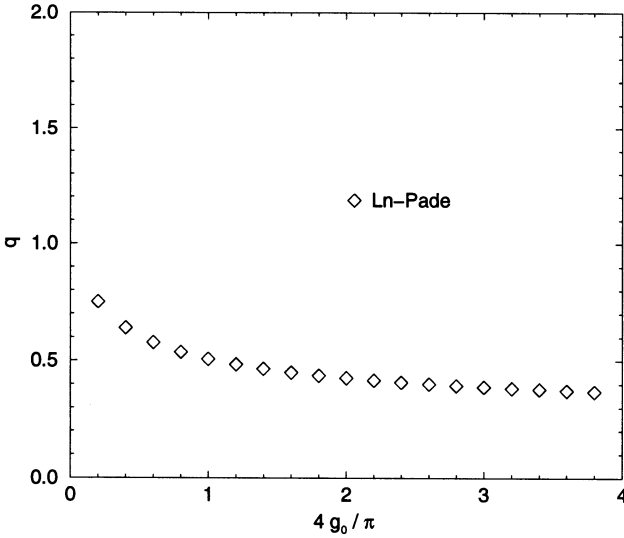
Fig. 6. Analysis of the series expansion of the model specific-heat using the Ln-Padé method.

heat becomes apparently weaker for stronger disorder. This may be taken as an indication that the true singularity is of the log–log type (7) as predicted by both, DD and SSL. A recent numerical study [21] for $J_2/J_1 = 4$ using transfer-matrix methods also observed a behavior in between log and log–log type. These findings are in contradiction to the claim [45,46] for a slightly different disordered system (quenched, random site-dilution) that the specific heat stays finite at T_c, as theoretically suggested in Ref. [10] (see also Refs. [47–49]).

Again we have tried to justify our interpretation by considering a model function,

$$C_{\text{model}} = \frac{1}{g_0} \ln \left[1 + \frac{4g_0}{\pi} \ln(1/\dot{t}) \right] . \tag{26}$$

By applying precisely the same type of analysis to the series expansion of the model specific-heat we obtained the results displayed in Fig. 6, which show qualitatively the same trend of decreasing q as a function of J_2/J_1 as the data in Fig. 5.

7. Discussion

The main results of our high-temperature series analysis are shown in Fig. 2 which provide at least for strong disorder (large J_2/J_1) compelling evidence that the singularity of the susceptibility is properly described by $\chi \propto t^{-7/4} |\ln t|^p$, with $p = 7/8 = 0.875$, as theoretically predicted by SSL [4–9]. The analysis of the model susceptibility (24) in Figs. 3 and 4 clearly shows that the apparent variation of p with the strength of disorder is an artifact caused by the truncation of the series expansions at a finite order.

We, therefore, emphasize that the apparent crossover from weak to strong disorder does *not* imply that the universality class of the random-bond Ising model changes continuously with the strength of disorder.

Let us finally make a few comments on previous Monte Carlo simulations of this model on large but finite square lattices. With the finite-size scaling analysis of Refs. [11–15,20], it is conceptually impossible to detect the multiplicative logarithmic correction of the SSL prediction (9). The reason is that the SSL theory also predicts a logarithmic correction for the scaling behavior of the correlation length, $\xi \propto t^{-1}|\ln t|^{1/2}$. In the finite-size scaling behavior the two logarithms thus cancel and one ends up with a pure power-law, $\chi \propto L^{\gamma/\nu} = L^{7/4}$, where L is the linear lattice size. Thus, *only* the SSL prediction for γ/ν can be tested in finite-size scaling analyses. Wang et al. [12,13] obtained for $J_2/J_1 = 4$ and 10 an estimate of $\gamma/\nu = 1.7507 \pm 0.0014$, and also the results of Reis et al. [20] at $J_2/J_1 = 2, 4$ and 10 are consistent with $\gamma/\nu = 1.75$. Among the two alternatives, the theories of DD and SSL, these estimates thus provide evidence in favor of SSL. Notice, however, that a numerical estimate of $\gamma/\nu \approx 1.75$ would also be expected for the *pure* two-dimensional Ising model. For the specific heat the situation is conceptually clearer. Here the theoretically expected scaling behavior (7), as predicted by both, DD and SSL, translates into a double-logarithmic finite-size scaling behavior, $C = C_0 + C_1 \ln(1 + b \ln L)$, which is different from that of the pure case where $C = C_0 + \ln L$. In the numerical work of Wang et al. [12,13], employing lattice sizes up to $L = 600$, this difference in the asymptotic behavior is clearly observed for $J_2/J_1 = 10$, while for $J_2/J_1 = 4$ the behavior is in between log and log–log type, similar to the findings reported in a recent transfer-matrix study [21] for the same coupling-constant ratio. For the specific heat these latest finite-size scaling analyses are thus about as conclusive as our series analyses in Fig. 5.

Another set of numerical data that can discriminate between the predictions of DD and SSL comes from direct simulations of the temperature dependence of the magnetization m and of the susceptibility χ for $J_2/J_1 = 4$ [13,16]. Assuming in the analysis a pure power law with an *effective* exponent (i.e., ignoring the logarithmic correction), one observes an overshooting of the effective exponent to values larger than the prediction by SSL. As discussed above (recall Eq. (22)) this may be taken as an indication of a multiplicative logarithmic correction term. For example, Talapov and Shchur [16] obtained for $J_2/J_1 = 4$ from least-squares fits to $\chi \propto t^{-\gamma_{\text{eff}}}$ an effective exponent of $\gamma_{\text{eff}} \approx 7/4 + 0.135 = 1.885$. This value is quite close to our series estimate of $\gamma_{\text{eff}} = 2.019 \pm 0.024$ for $J_2/J_1 = 4$, if the pure power-law ansatz is used (cf. Fig. 1). Wang et al. [13] furthermore confirmed that their data is compatible with the SSL ansatz, $\chi(t) = \chi_0 t^{-7/4}(1 + at)[1 + b \ln(1/t)]^{7/8}$, supplemented by a correction to scaling term $(1 + at)$ (and similarly for m; for a recent confirmation, see Ref. [19]). In these fits both exponents are kept fixed at their predicted values, and χ_0, a, and b are free parameters. In contrast to our series analysis, however, no quantitative estimates of the exponent of the logarithmic correction have been reported in Ref. [13]. While the simulation results certainly indicate that among the two conflicting theories of DD and SSL, the SSL prediction is more likely to be correct, it is still fair to conclude

that also this set of simulations has not yet unambiguously identified the multiplicative logarithmic correction term.

Monte Carlo simulations of systems with quenched, random disorder require an enormous amount of computing time because many realisations have to be simulated for the quenched average. For this reason it is hardly possible to scan a whole parameter range. Using high-temperature series expansions, on the other hand, one can obtain closed expressions in several parameters (such as the dimension d, x, J_2/J_1, ...) up to a certain order in the inverse temperature $\beta = 1/k_B T$. Here the infinite-volume limit is always implied and the quenched, random disorder can be treated exactly. By analyzing the resulting series, the critical behavior of the random-bond system can hence in principle be monitored as a continuous function of several parameters. This is a big advantage over Monte Carlo simulations which usually can only yield a rather small parameter range in one set of simulations. The caveat of the series-expansion approach is that the available series expansions for the random-bond Ising model are still relatively short (at any rate much shorter than for pure systems). This introduces systematic errors of the resulting estimates for critical exponents which are difficult to control. The obvious way out is trying to extend the series expansions as far as possible. This, however, would be extremely cumbersome since the number of algebraic manipulations necessary to calculate the series coefficients blows up dramatically with the order of the series (usually at least exponentially) and, therefore, has to be left for future work.

Acknowledgements

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