

Finite Size Scaling, Fisher Zeroes and $\mathcal{N}=4$ Super Yang-Mills

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We investigate critical slowing down in the local updating continuous-time Quantum Monte Carlo method by relating the finite size scaling of Fisher Zeroes to the dynamically generated gap, through the scaling of their respective critical exponents. As we comment, the nonlinear sigma model representation derived through the hamiltonian of our lattice spin model can also be used to give a effective treatment of planar anomalous dimensions in $\mathcal{N}=4$ SYM. We present scaling arguments from our FSS analysis to discuss quantum corrections and recent 2-loop results, and further comment on the prospects of extending this approach for calculating higher twist parton distributions.

1. VBS PHASE TRANSITIONS

We investigate the critical behavior of Valence-Bond-Solid (VBS) states in quantum spin chains by means of Quantum Monte Carlo (QMC) simulation using the continuous-time loop cluster algorithm [1]. Following *Haldane's conjecture* a variety of exotic magnetic phenomena can be attributed to the formation of ground states with an energy-gap in integer spin systems at low temperatures. These gapped states are predicted to have transitions with a massless excitation between phases, and can be effectively expressed in terms of a VBS state (spin-singlet) picture. The *Lieb-Schultz-Mattis argument* to this conjecture has led to the recent introduction of a string exact order parameter to characterise these VBS transitions. We investigate the Finite Size Scaling (FSS) properties of a generalised *twist* order parameter for a periodic mixed-spin chain [2] with a unit cell of the form $(1, 1, \frac{1}{2}, \frac{1}{2})$. We also determine the FSS properties of the complex-temperature (Fisher) zeroes of the partition function [3], evaluated in a new scheme through knowledge of the QMC transfer matrix. This enables us to separate pseudocritical and critical point scaling behavior relating to the correlation length exponent and also to locate the VBS state transition points through an independent and expeditious means. Leading corrections to

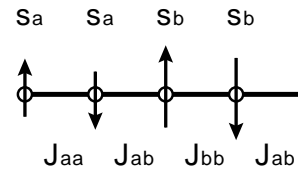


Figure 1. Periodic mixed spin cell, [4].

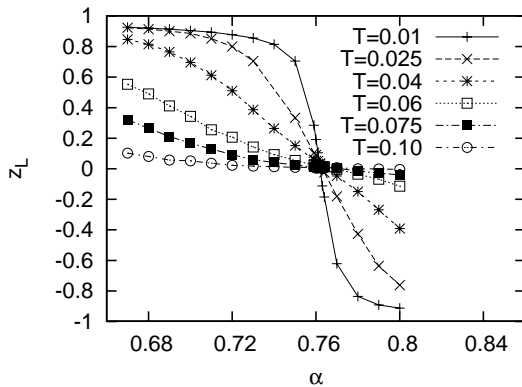
the FSS of the zeroes are also known exactly for comparable models such as the 2D Ising model with Brascamp-Kunz boundary value conditions [5]. We compare the FSS of these indicators to evaluate the critical exponents for the VBS transitions, comparing with nonlinear σ -model predictions.

The Hamiltonian for the AHFM two-spin chain with spins \mathbf{S}^a and \mathbf{S}^b of period-4 is given as,

$$H = \sum_{j=1}^{N/4} (J_{aa} \mathbf{S}_{4j+1}^a \cdot \mathbf{S}_{4j+2}^a + J_{ab} \mathbf{S}_{4j+2}^a \cdot \mathbf{S}_{4j+3}^b + J_{bb} \mathbf{S}_{4j+3}^b \cdot \mathbf{S}_{4j+4}^b + J_{ab} \mathbf{S}_{4j+4}^b \cdot \mathbf{S}_{4j+5}^a) \quad (1)$$

with $J_{aa}=J_{bb}=1$ and the coupling anisotropy $\alpha = J_{ab}/J_{aa}$.

We anticipate competing low-temperature VBS dimer gap states from nonlinear sigma model treatment [4], separated via a quantum phase

Figure 2. Twist order parameter, z_L .

transition at critical anisotropy at α_c . We use a Lieb-Schultz-Mattis extension *twist-operator* on the groundstate ($|\Psi\rangle = U|\Psi_0\rangle$) [6], as an exact order parameter for the VBS states to signal this transition,

$$\mathcal{O}_{\text{string}} = - \lim_{|k-l| \rightarrow \infty} \langle \Psi_0 | S_k \exp \left[i\pi \sum_{j=k+1}^{l-1} S_j \right] S_l | \Psi_0 \rangle, \quad (2)$$

$$U \equiv \exp \left[i \frac{2\pi}{L} \sum_{j=1}^L j S_j^z \right], \quad z_L \equiv \langle \Psi_0 | \Psi \rangle, \quad (3)$$

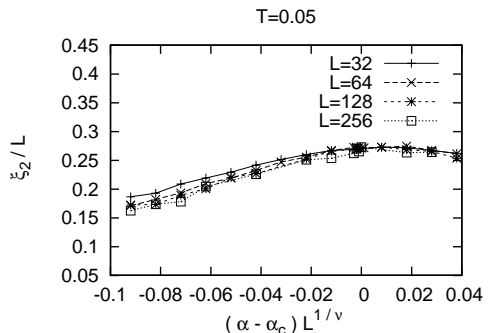
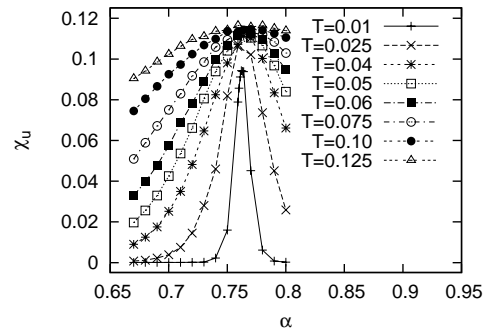
Preliminary indications are of a low-temperature 2nd-order transition line at a (constant) α_c with a critical endpoint at $\beta_c = 0.09$ followed by a crossover, Fig.2 and Fig.3.

2. QMC TRANSFER MATRIX

Following the Quantum Monte Carlo *improved-estimator* definitions for conventional thermodynamic (2nd derivative of free energy) observables [7] such as the uniform χ_u and the second moment definition of correlation length ξ_2 ,

$$C(\mathbf{r}) = \frac{1}{\beta^2} \int_0^\beta d\tau d\tau' \langle \hat{S}_i(\tau) \hat{S}_{i+\mathbf{r}}(\tau') \rangle, \quad (4)$$

$$\chi(\mathbf{q}) = \beta \sum_{\mathbf{r}} e^{i\mathbf{q}\cdot\mathbf{r}} C(\mathbf{r}) : \quad \chi_u = \chi(\mathbf{q}=0), \quad (5)$$

Figure 3. Uniform susceptibility χ_u , and ξ_2/L .

$$\xi_2 = \frac{L}{2\pi} \sqrt{\frac{\chi(\pi, \pi)}{\chi(\pi + 2\pi/L, \pi)} - 1}, \quad (6)$$

we introduce a Fisher zeros *improved-estimator* polynomial expansion in the coupling J_{ab} to determine independently the FSS properties of α_c . We plot the Fisher zeros in the complex J_{ab}^2 plane in Fig.4, noting the pinching of the real axis corresponding to the discontinuities in the partition function that in the thermodynamic limit give α_c . The *inner* edge-singularity gives the quantum phase transition (crossover) point α_c , and also a clear indication of finite size effects where the zeroes solution reflected from the QMC Euclidean-time boundary conditions on our smallest volume ($L = 64$) are lost. The *outer* edge-singularity gives us a new means to study the criticality of the Spontaneous Symmetry Breaking associated with the dimensional reduction of the QMC approach (akin to the 1D Ising model, where a nonzero magnetisation gives a non-analyticity in zeroes on the unit circle). Note the volume-dependent deviation from unity for imaginary valued ($J_{ab}^2 < 0$) critical couplings with relation to

the *twist-operator*. Although evidently there is no VBS quantum interference for imaginary couplings, the action still acquires a nonzero phase expectation, scaling with finite L [8].

3. LIGHT CONE & INTEGRABILITY

The correspondence of spin chains with light cone field theoretic treatments has been noted and studied via exact techniques such as the TBA [9][10][11]. For an explicit determination of twist-3 operators in QCD a further knowledge of the evolution at finite N_c is required, however [12][13]. Our Fisher zeroes scaling (under investigation) will now allow us to quantify this scaling in concert with the RG physics of the VBS states.

REFERENCES

1. B.B. Beard and U.-J. Wiese, Nucl. Phys. Proc. Suppl. **53**, 838 (1997) ; U.-J. Wiese and H.P. Ying, Z. Phys. B **93**, 147 (1994).
2. Z. Xu, J. Dai, H. Ying and B. Zheng, Phys. Rev. B **67**, 214426 (2003).
3. M.E. Fisher, in *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, 1965), Vol. 7c ; C. Itzykson, R. B. Pearson and J. B. Zuber, Nucl. Phys. B **220** [FS8], 415 (1983).
4. K. Takano, Phys. Rev. B **61**, 8863 (2000).
5. W. Janke and R. Kenna, Phys. Rev. B **65**, 064110 (2002).
6. M. Nakamura and S. Todo, Phys. Rev. Lett. **89**, 077204 (2002) ; R. Resta and S. Sorella, Phys. Rev. Lett. **82**, 370 (1999).
7. A. Cuccoli, T. Roscilde, V. Tognetti, R. Vaia and P. Verrucchi, Phys. Rev. B **67**, 104414 (2003).
8. P.R. Crompton, Nucl. Phys. Proc. Suppl. **119**, 550 (2003).
9. S. Dalley, Phys. Lett. B **334**, 61 (1994).
10. M. Kruczenski, A.V. Ryzhov and A. A. Tseytlin, Nucl. Phys. B **692**, 3 (2004).
11. N. Beisert, Nucl. Phys. B **682**, 487 (2004).
12. L.D. Faddeev and G.P. Korchemsky, Phys. Lett. B **342**, 311 (1995).
13. S.E. Derkachov, G.P. Korchemsky and A.N. Manashov, Nucl. Phys. B **566**, 203 (2000).

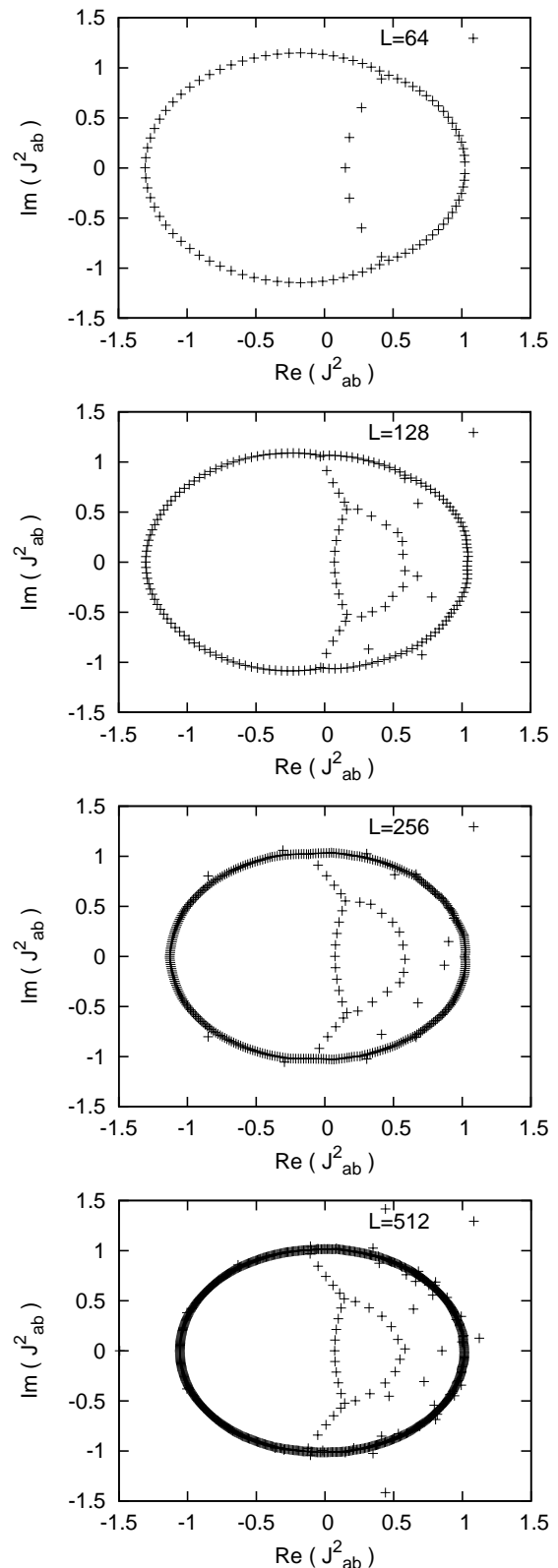


Figure 4. Fisher zeroes in the complex J_{ab}^2 plane.