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Algorithmic tools for simulations of vertex models on random graphs* †

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We consider the coupling of ice-type vertex models to random, planar ϕ^4 quantum-gravity graphs. The well-established techniques for the simulation of dynamical triangulations and their dual ϕ^3 graphs are suitably adapted to the case of four-valent graphs. These methods are combined with a formulation of the *loop algorithm* for the simulation of the vertex model matter part. We present a preliminary analysis of the dynamical scaling behaviour of the combined algorithm for the case of the 6-vertex model coupled to quantum gravity.

1. INTRODUCTION

Vertex models on regular lattices exhibit an exceptionally rich phase structure, including lines of first- and second-order phase transitions as well as critical and multi-critical points [1]. A number of classic lattice models such as the Ising and Potts models and graph-colouring problems correspond to limiting cases of the 8-vertex model. If vertex models are coupled to non-perturbative quantum gravity in the form of random planar ϕ^4 graphs, corresponding to the dual lattices of dynamical "quadrangulations", a richness of the phase diagram similar to the case of the square lattice can be expected. Recently, the use of matrix model methods led to a conjecture for the behaviour of a limiting case of the 6-vertex model, the F model [2]; for a special slice of the 8-vertex model, see [3].

We present the numerical setup of a Monte Carlo (MC) simulation scheme for the analysis of 6- and 8-vertex models coupled to planar ϕ^4 graphs. Apart from verifying the conjectured results for the F model, this type of simulations can clarify the properties of the more general cases not yet covered by the matrix model approach.

2. DYNAMICAL ϕ^4 GRAPHS

While MC simulations of dynamical *triangulations* [4] have been used successfully for more than a decade, dynamical *quadrangulations* and the dual ϕ^4 graphs have received much less attention. First, concerning the amount of singular contributions allowed in the ϕ^4 graphs we distinguish a hierarchy of four different ensembles of graphs:

- *singular*: no restrictions
- *restricted singular*: no seagull contributions

- *regular*: in addition no self-energies

- *strict*: in addition no double bonds

Here, the excluded contributions not only include the depicted local singularities, but also "dressed" seagulls, i.e. one-point subgraphs, and "dressed" self-energies, i.e. two-point subgraphs; the double bonds are purely local contributions.

For the case of dynamical triangulations and their dual ϕ^3 graphs, it has been shown [5] that the so-called Pachner or (k,l) moves constitute a set of ergodic updates for a MC process. In two dimensions, these moves are readily generalized to

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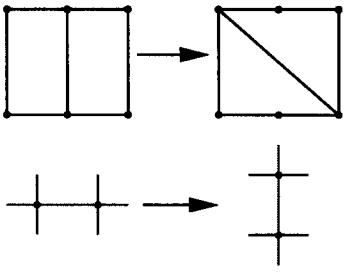
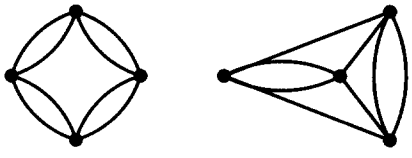
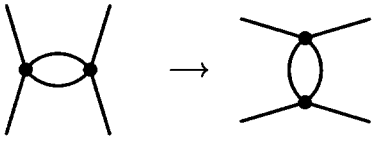
the case of ϕ^4 graphs; the volume preserving $(2,2)$ move for quadrangulations (top) and ϕ^4 graphs (bottom) is given by:



The original $(2,2)$ or flip move for ϕ^3 graphs can be shown to be ergodic within the class of combinatorial manifolds, i.e., the equivalent of the *strict* ensemble for the ϕ^4 case above [5], and is commonly believed to be ergodic for the singular cases also. However, in all but the *strict* ensemble in the ϕ^4 case, there are possible configurations like the following "ring" and "center" diagrams,



which are in all but the *singular* ensembles obviously not connected to each other by a $(2,2)$ flip move. Thus, the moves depicted above are not ergodic for the non-*strict* ensembles. However, we find that augmenting the $(2,2)$ move with a *two-link flip* of the form



ensures ergodicity for the non-*strict* ensembles also.

3. VERTEX MODELS

The vertex configurations of the 6-vertex model depicted in Fig. 1 carry energies ϵ_i, $i = 1,\ldots,6$; the overall arrow reversal symmetry is reflected by $\epsilon_i = \epsilon_{i+1}$, $i = 1,3,5$. Thus, the Boltzmann weights of the model are given by [1]

$$a = e^{-K\epsilon_1}, \quad b = e^{-K\epsilon_3}, \quad c = e^{-K\epsilon_5}, \tag{1}$$

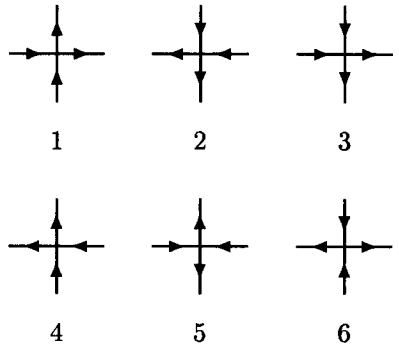


Figure 1. The 6-vertex model configurations.

where K denotes the thermal coupling. Obviously, on a random graph one has to impose the additional restriction $a = b$, since the vertices 1 and 3 resp. 2 and 4 are related to each other only by rotations of $\pi/2$. This choice of weights corresponds to the F model of statistical mechanics; we choose $a = b = \exp(-K)$ and $c = 1$. On the square lattice this model undergoes a continuous phase transition to an anti-ferroelectrically ordered phase at $K_c = \ln 2$ [1].

For the proper definition of an order parameter for the anti-ferroelectric transition of the F model on a random lattice one needs a *two-colouring* of the *dual* lattice of the ϕ^4 graph, i.e. the corresponding dynamical quadrangulation, leading to

$$M = \sum_{l\in\mathbb{L}} \sum_{b_l\in l} C(l)\, v(b_l), \tag{2}$$

where $\mathbb{L}$ denotes the set of faces or minimal loops on the ϕ^4 graphs, b_l are the bonds belonging to loop l, $C(l) \in \{+1,-1\}$ is the "colour" of loop l and $v(b_l)$ the arrow direction of a given vertex configuration on bond l relative to a given, say anti-clockwise, trespassing of the loops.

4. DYNAMICAL SCALING

Combining the described update moves for the random lattice with an implementation of the *loop algorithm* [6], a non-local algorithm of the cluster type, for the vertex model part, we consider integrated autocorrelation times for the total update measured in units of sweeps consisting

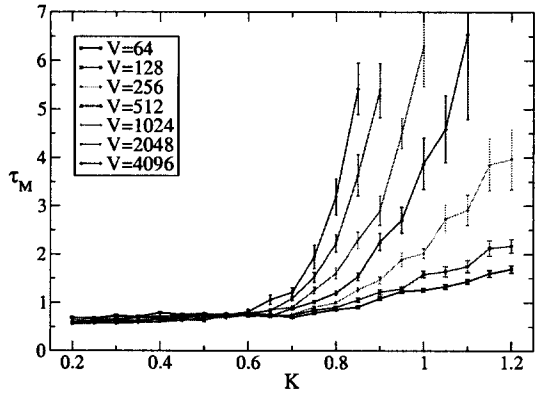


Figure 2. Integrated autocorrelation times for the magnetization of the 6-vertex model on planar ϕ^4 graphs of the *regular* ensemble.

of 10 sweeps of link flip moves and one multi-cluster update of the vertex model. As properties of the vertex model part we measure the total system energy E defined from (1) and the magnetization (2). As generic graph property we consider the mean square extent $\langle r^2 \rangle$ of the lattice, which is known to be the slowest mode of relaxation in most cases. Integrated autocorrelation times for all three quantities are determined from a direct integration of the measured autocorrelation function as well as a combined binning/jackknife technique [7], the latter of which is also used for the determination of statistical errors in both cases.

The dynamical behaviour of the algorithm is demonstrated in Fig. 2 for the magnetization; autocorrelation times diverge exponentially in the low-temperature phase as expected. For the *regular* ensemble, the finite-size scaling of τ_M at the conjectured critical point $K_c = \ln 2$ [2] yields a dynamical critical exponent of

$$z_M/d_H = 0.155(27), \tag{3}$$

where d_H denotes the (unknown) Hausdorff dimension of the lattice, and for the mean square extent $\langle r^2 \rangle$ we obtain

$$z_{\langle r^2 \rangle}/d_H = 0.724(58). \tag{4}$$

This indicates that these two observables pick up dynamical modes which are "orthogonal" to each other, yielding clearly different dynamical critical exponents. The total system energy, on the other hand, yields very small and almost constant autocorrelation times for all temperatures. This is connected to the sub-lattice structure of the ordered state, which leads to strongly anti-correlated sub-lattice energies as observed for vertex models on regular lattices [6]. On random lattices, however, the concept of sub-lattice energies is maldefined, such that we cannot consider the dynamical scaling of such quantities. For the case of the *strict* ensemble, autocorrelation times for all quantities are strongly enhanced in amplitude due to the strong restrictions for the flip moves. The critical exponents z/d_H, however, are compatible between ensembles as expected.

5. CONCLUSIONS

With proper care a generalization of the Pachner moves to dynamical ϕ^4 graphs and the definition of a vertex model order parameter are possible. The large dynamical exponent of the lattice part, however, calls for the adaption and application of non-local update schemes.

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