

**ON THE CONTINUUM LIMIT OF THE ISING MODEL  
COUPLED TO REGGE GRAVITY<sup>1</sup>**

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The Regge Calculus is a powerful method to approximate a continuous manifold by a simplicial lattice, keeping the connectivities of the underlying lattice fixed and taking the edge lengths as degrees of freedom. The Discrete Regge Model employed in this work limits the choice of the link lengths to a finite number. For both theories there seems to be a discontinuous phase transition at positive gravitational coupling and a smooth transition at negative coupling. To construct a simple theory of quantum gravity with matter fields, we couple the four-dimensional Discrete Regge Model to Ising spins. We examined the phase transition of the spin system and the associated critical exponents. The results are obtained from finite-size scaling analyses of Monte Carlo simulations. We find consistency with the mean-field critical behavior of the Ising model on a static four-dimensional lattice. A main concern in lattice field theories is the existence of a continuum limit which requires the existence of a continuous phase transition. The second-order transition of the Ising model together with the recently conjectured second-order transition of the Regge skeleton at negative gravity coupling could be such a candidate.

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## 1 Introduction

We study spin systems coupled to fluctuating manifolds as a simple example for matter fields coupled to Euclidean quantum gravity. To describe the gravity sector we use the Discrete Regge Model [1–3] which is both structurally and computationally much simpler than the Standard Regge Calculus [4] with continuous link lengths. Here numerical simulations can be done more efficiently by implementing look-up tables and using the heat-bath algorithm. In the actual computations we took the squared link lengths as  $q_{ij} \equiv q_l = b_l(1 + \epsilon\sigma_l)$  with  $\sigma_l = \pm 1$  and  $\epsilon = 0.0875$ . Since a four-dimensional Regge skeleton with equilateral simplices cannot be embedded in flat space,  $b_l$  takes different values depending on the type of the edge  $l$ . In particular

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$b_l = 1, 2, 3, 4$  for edges, face diagonals, body diagonals, and the hyperbody diagonal of a hypercube. The restriction of the edge lengths to just two values was carefully examined in 2D where an interpolation from  $\sigma_l \in Z_2$  to  $Z_\infty$  was performed [5]. It turned out that the phase transition with respect to the cosmological constant is universal. This was tested for pure gravity in 2D and is expected to be the case also in 4D.

## 2 Model and observables

We investigated the partition function

$$Z = \sum_{\{s\}} \int D[q] \exp[-I(q) - KE(q, s)], \quad (1)$$

where  $I(q)$  is the gravitational action,

$$I(q) = -\beta_g \sum_t A_t \delta_t + \lambda \sum_i V_i. \quad (2)$$

The first sum runs over all products of triangle areas  $A_t$  times corresponding deficit angles  $\delta_t$  weighted by the gravitational coupling  $\beta_g$ . The second sum extends over the volumes  $V_i$  of the 4-simplices of the lattice and allows one together with the cosmological constant  $\lambda$  to set an overall scale in the action. The energy of Ising spins  $s_i \in Z_2$ ,

$$E(q, s) = \frac{1}{2} \sum_{\langle ij \rangle} A_{ij} \frac{(s_i - s_j)^2}{q_{ij}}, \quad (3)$$

is defined as in two dimensions [6], with the barycentric area  $A_{ij}$  associated with a link  $l_{ij}$ ,  $A_{ij} = \sum_{t \supset l_{ij}} \frac{1}{3} A_t$ . We chose the simple uniform measure as in the pure gravity simulations [3],  $D[q] = \prod_l dq_l \mathcal{F}(q_l)$ . The function  $\mathcal{F}$  ensures that only Euclidean link configurations are taken into account which is trivially satisfied for the Discrete Regge Model.

For every Monte Carlo simulation run we recorded the time series of the energy density  $e = E/N_0$  and the magnetization density  $m = \sum_i s_i/N_0$ , with the lattice size  $N_0 = L^4$ . To obtain results for the various observables  $\mathcal{O}$  at values of the spin coupling  $K$  in an interval around the simulation point  $K_0$ , we applied standard reweighting techniques [7].

With the help of the time series we compute the specific heat,  $C(K) = K^2 N_0 (\langle e^2 \rangle - \langle e \rangle^2)$ , the (finite lattice) susceptibility,  $\chi(K) = N_0 (\langle m^2 \rangle - \langle |m| \rangle^2)$ , and various derivatives of the magnetization,  $d\langle |m| \rangle/dK$ ,  $d\ln\langle |m| \rangle/dK$ , and  $d\ln\langle m^2 \rangle/dK$ . All these quantities exhibit in the infinite-volume limit singularities at the critical coupling  $K_c$  which are shifted and rounded in finite systems. We further analyzed the Binder parameter,  $U_L(K) = 1 - \langle m^4 \rangle / (3\langle m^2 \rangle^2)$ .

## 3 Simulation results

In four dimensions, after initial discussions [8–10] it is generally accepted that the critical properties of the Ising model on a static lattice are given by mean-field theory, with logarithmic corrections. The finite-size scaling (FSS) formulas can be written as [11, 12]

$$C \propto (\log L)^{\frac{1}{3}}, \quad (4)$$

$$\chi \propto (L(\log L)^{\frac{1}{4}})^{\gamma/\nu}, \quad (5)$$

$$\xi \propto L(\log L)^{\frac{1}{4}}, \quad (6)$$

$$K_c(\infty) - K_c(L) \propto (L(\log L)^{\frac{1}{12}})^{-1/\nu}, \quad (7)$$

where  $\xi$  is the correlation length and the critical exponents of mean-field theory are  $\alpha = 0$ ,  $\beta = 1/2$ ,  $\gamma = 1$ , and  $\nu = 1/2$ . The gravitational degrees of freedom of the partition function (1) were updated with the heat-bath algorithm. For the Ising spins we employed the single-cluster algorithm [13]. Between measurements we performed  $n = 10$  Monte Carlo steps consisting of one lattice sweep to update the squared link lengths  $q_{ij}$  followed by two single-cluster flips to update the spins  $s_i$ . The simulations were done for cosmological constant  $\lambda = 0$  and gravitational coupling  $\beta_g = -4.665$ . At this  $\beta_g$ -value a second-order phase transition of the pure Discrete Regge Model was conjectured [3], see Fig. 1. Together with an eventual second-order transition of the Ising model part, this transition could thus be a candidate for a possible continuum limit.

The lattice topology is given by triangulated tori of size  $N_0 = L^4$  with  $L = 3$  up to 10. From short test runs we estimated the location of the phase transition of the spin model and set the spin coupling  $K_0 = 0.024 \approx K_c$  in the long runs. After an initial equilibration time we took about 100 000 measurements for each lattice size. Analyzing the time series we found integrated auto-correlation times for the energy and the magnetization in the range of unity for all lattice sizes. The statistical errors were obtained by the standard Jack-knife method using 50 blocks. Applying the reweighting technique we first determined the maxima of  $C$ ,  $\chi$ ,  $d\langle|m|\rangle/dK$ ,  $d\ln\langle|m|\rangle/dK$ , and  $d\ln\langle|m^2|\rangle/dK$ . The locations of the maxima provide us with five sequences of pseudo-transition points  $K_{\max}(L)$  for which the scaling variable  $x = (K_c - K_{\max}(L))(L(\log L)^{\frac{1}{12}})^{\frac{1}{\nu}}$  should be constant. Using this fact we then have several possibilities to extract the critical exponent  $\nu$  from (linear) least-square fits of the FSS ansatz with logarithmic corrections (7),

$$dU_L/dK \cong (L(\log L)^{\frac{1}{12}})^{1/\nu} f_0(x), \quad (8)$$

$$d\ln\langle|m|^p\rangle/dK \cong (L(\log L)^{\frac{1}{12}})^{1/\nu} f_p(x), \quad (9)$$

to the data at the various  $K_{\max}(L)$  sequences. The exponents  $1/\nu$  resulting from fits using the data for  $L = 4 - 10$  are collected in Tab. 1, where  $Q$  denotes the standard goodness-of-fit parameter. For our simulations all exponent estimates with the logarithmic corrections and consequently also their weighted average  $1/\nu = 2.028(7)$  are in agreement with the mean-field value  $1/\nu = 2$ . We further performed fits with a naive power-law FSS ansatz neglecting the multiplicative logarithmic corrections. Here one also gets an estimate for  $1/\nu$  close to the mean-field value, but clearly separated from it.

Assuming therefore  $\nu = 1/2$  we can obtain estimates for  $K_c$  from linear least-square fits to the scaling behavior of the various  $K_{\max}$  sequences, as shown in Fig. 2. Using the fits with  $L \geq 4$ , the combined estimate from the five sequences leads to  $K_c = 0.02464(4)$ .

Knowing the critical coupling we may reconfirm our estimates of  $1/\nu$  by evaluating the above quantities at  $K_c$ . As can be inspected in Tab. 1, the statistical errors of the FSS fits at  $K_c$  are similar to those using the  $K_{\max}$  sequences. However, here we also have to take into account the

fit type	$1/\nu$	$Q$
$dU/dK$ at $K_{\max}^C$	1.980(17)	0.70
$d\ln\langle m \rangle/dK$ at $K_{\inf}^{\ln\langle m \rangle}$	2.032(10)	0.59
$d\ln\langle m^2\rangle/dK$ at $K_{\inf}^{\ln\langle m^2\rangle}$	2.038(10)	0.55
weighted average	2.028(7)	
$dU/dK$ at $K_c$	1.981(17)[13]	0.70
$d\ln\langle m \rangle/dK$ at $K_c$	2.027(9)[2]	0.95
$d\ln\langle m^2\rangle/dK$ at $K_c$	2.034(9)[2]	0.85
weighted average	2.025(6)	
overall average	2.026(5)	

Tab. 1. Fit results for  $1/\nu$  in the range  $L = 4 - 10$  with a power-law ansatz with logarithmic corrections.

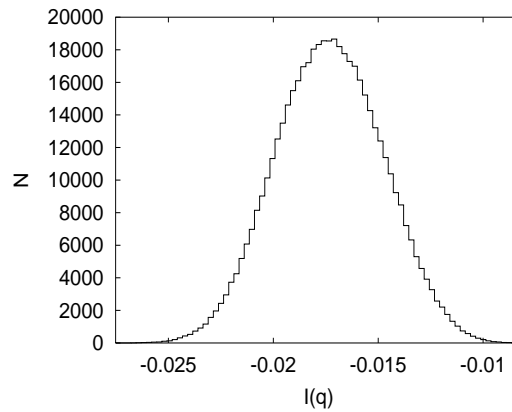


Fig. 1. Histogram of the gravitational action (2) from a simulation [3] of the pure Discrete Regge Model on a  $4^4$  lattice at  $\beta_g = -4.665$ .

uncertainty in our estimate of  $K_c$ . This error is computed by repeating the fits at  $K_c \pm \Delta K_c$  and indicated in Tab. 1 by the numbers in square brackets. In the computation of the weighted average we assume the two types of errors to be independent. As a result of this combined analysis we obtain strong evidence that the exponent  $\nu$  agrees with the mean-field value of  $\nu = 1/2$ .

To extract the critical exponent ratio  $\gamma/\nu$  we use the scaling (5) of the susceptibility  $\chi$  at its maximum as well as at  $K_c$ , yielding in the range  $L = 4 - 10$  estimates of  $\gamma/\nu = 2.039(9)$  ( $Q = 0.42$ ) and  $\gamma/\nu = 2.036(7)[4]$  ( $Q = 0.85$ ), respectively. These estimates for  $\gamma/\nu$  are compatible with the mean-field value of  $\gamma/\nu = 2$ . In Fig. 3 this is demonstrated graphically by comparing the scaling of  $\chi_{\max}$  with a constrained one-parameter fit of the form  $\chi_{\max} = c(L(\log L)^{\frac{1}{4}})^2$  with  $c = 4.006(10)$  ( $Q = 0.17$ ,  $L \geq 6$ ).

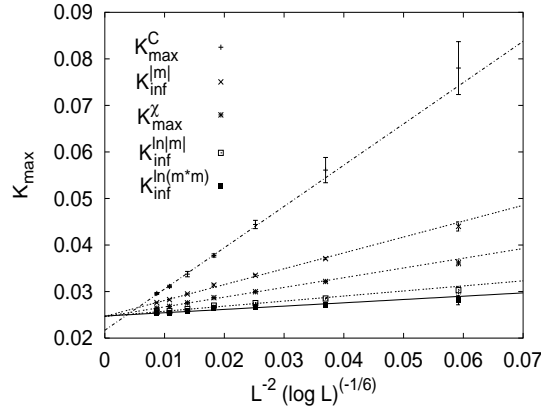


Fig. 2. FSS extrapolations of pseudo-transition points  $K_{\max}$  vs.  $(L(\log L)^{\frac{1}{2}})^{-1/\nu}$ , assuming  $\nu = 1/2$ . The error-weighted average of extrapolations to infinite size yields  $K_c = 0.02464(4)$ .

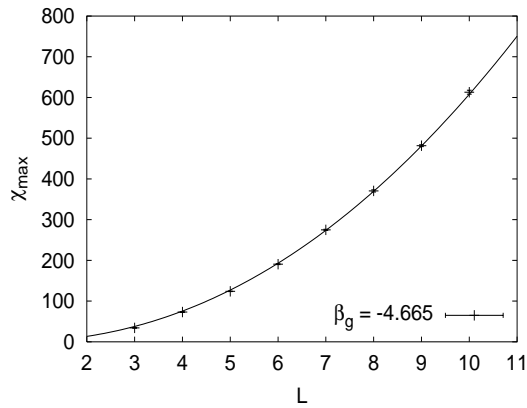


Fig. 3. FSS of the susceptibility maxima  $\chi_{\max}$ . The exponent entering the curve is set to the mean-field value  $\gamma/\nu = 2$  for regular static lattices.

#### 4 Conclusions

We have performed a study of the Ising model coupled to fluctuating manifolds via Regge Calculus. Analyzing the Discrete Regge Model with two permissible edge lengths it turns out that the Ising transition shows the expected logarithmic corrections to the mean-field theory. We have also studied the pure Ising model on a rigid lattice [14] without presenting the results in this short note. The critical exponents of the phase transition of the Ising spins on a static lattice as well as on a discrete Regge skeleton are both consistent with the exponents of mean-field theory,  $\alpha = 0$ ,  $\beta = 1/2$ ,  $\gamma = 1$ , and  $\nu = 1/2$ . In summary, from our comparative analysis with uniform computer codes we conclude that the phase transition of the Ising spin model coupled to a discrete Regge skeleton exhibits the same critical exponents and the same logarithmic corrections as on a

static lattice. With respect to future work, we note that an investigation of the critical exponents of the pure Discrete Regge Model is interesting by its own. If it could be substantiated that also the gravity part of the theory undergoes a continuous transition, this would ensure the existence of a continuum limit which has been under intense discussion in recent years [15, 16].

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